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Research Paper

Permanence and almost periodic solution for a competition and cooperation model of enterprise cluster with feedback controls

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ABSTRACT
In this article, based on population ecology theory, we present a competition and cooperation system of the enterprise cluster with feedback controls. Based on the theory of calculus on time scales, by using the properties of almost periodic functions and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the permanence and the existence of a unique globally attractive positive almost periodic solution of the system are obtained.
KEYWORDS: *Permanence; Almost periodic solution; Asymptotic stability; Time scale.*

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I. INTRODUCTION

Enterprise cluster refers to the concentration of similar or related enterprises in a specific area, which form fixed economic outputs and have certain economic influence on outsides [1]. After a large number of observations, it is found that there is a similarity between the enterprise cluster and the ecological population system. Recently, some researchers have presented some models about enterprise clusters based on ecology theory, which arouse growing interest in applying the methods of ecology and dynamic system theory to study enterprise clusters, for example [2-8] and references cited therein. For an example, in [9], the authors considered the following competition and cooperation of enterprise cluster based on the ecosystem

$$\begin{cases} x_1'(t) = r_1 x_1(t)[1 - \frac{1}{K} x_1(t) - \frac{1}{K}\alpha(x_2(t) - b_2)^2], \\ x_2'(t) = r_2 x_2(t)[1 - \frac{1}{K} x_2(t) + \frac{1}{K}\beta(x_1(t) - b_1)^2], \end{cases}$$

where $x_1(t), x_2(t)$ represent the outputs of enterprise A and enterprise B, respectively, $r_i, b_i, K, \alpha, \beta$ are positive constants, $i = 1, 2$. r_1, r_2 are the intrinsic growth rates, K denotes the carrying capacity of market under the natural conditions, α, β are the competitive power coefficients of the two enterprises, and b_1, b_2 are the initial productions of the enterprises, respectively.

Let $a_1 = \frac{r_1}{K}, a_2 = \frac{r_2}{K}, c_1 = \frac{\alpha}{K}, c_2 = \frac{\beta}{K}$, the system above becomes

$$\begin{cases} x_1'(t) = x_1(t)[r_1 - a_1 x_1(t) - c_1(x_2(t) - b_2)^2], \\ x_2'(t) = x_2(t)[r_2 - a_2 x_2(t) + c_2(x_1(t) - b_1)^2]. \end{cases}$$

In the real world, enterprises are continuously distributed by unpredictable forces which can result in changes in the economic parameters, if the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions; see, for example, [10-13]. However, there are seldom results on the existence of almost periodic solutions of enterprise cluster systems on time scales.

Motivated by above, in this paper, we propose a competition and cooperation model of enterprise cluster with feedback controls on time scales as follows:

$$\begin{cases} x_1^{\Delta}(t) = r_1(t) - a_1(t)\exp\{x_1(t)\} - c_1(t)(\exp\{x_2(t)\} - b_2)^2 - k_1(t)y_1(t), \\ x_2^{\Delta}(t) = r_2(t) - a_2(t)\exp\{x_2(t)\} + c_2(t)(\exp\{x_1(t)\} - b_1)^2 - k_2(t)y_2(t), \\ y_1^{\Delta}(t) = -h_1(t)y_1(t) + d_1(t)\exp\{x_1(t)\}, \\ y_2^{\Delta}(t) = -h_2(t)y_2(t) + d_2(t)\exp\{x_2(t)\}, \end{cases} \quad (1)$$

where $t \in \mathbb{T}, \mathbb{T}$ is an almost time scale; $x_1(t)$ and $x_2(t)$ denote the outputs of enterprises A and B in cluster respectively, $r_1(t)$ and $r_2(t)$ are their intrinsic growth rates at time $t, a_1(t)$ and $a_2(t)$ account for their self-regulation coefficients, $c_1(t)$ and $c_2(t)$ represent their contribution coefficients to the other, b_1, b_2 are the initial productions of the enterprises respectively; the latter two equations are control equations, $y_1(t)$ and $y_2(t)$ are feedback control variables.

In this paper, the time scale $\mathbb{T}$ considered is unbounded above, and for each interval $\mathbb{I}$ of $\mathbb{T}$, we denote by $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$. For convenience, we introduce the notation

$$f^u = \sup_{t\in\mathbb{T}} f(t),\ f^l = \inf_{t\in\mathbb{T}} f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that the coefficients $r_i(t), a_i(t), c_i(t), d_i(t), k_i(t), h_i(t), i = 1,2,$ are continuous, almost periodic functions, and satisfy

$$\min\{r_i^l, a_i^l, c_i^l, d_i^l, k_i^l, h_i^l\} > 0,\ \max\{r_i^u, a_i^u, c_i^u, d_i^u, k_i^u, h_i^u\} < +\infty, i = 1,2.$$

The initial condition of system (1.1) in the form

$$x_i(t_0) = x_{i0}, y_i(t_0) = y_{i0}, x_{i0} > 0, y_{i0} > 0, t_0 \in \mathbb{T}, i = 1,2. \quad (2)$$

The aim of this paper is, by using the properties of almost periodic functions and constructing a suitable Lyapunov functional, to obtain sufficient conditions for the permanence and the existence of a unique globally attractive positive almost periodic solution of system (1).

II. PRELIMINARIES

The basic theory of calculus on time scales, see [14].

Lemma 1 ([14]) If $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then

$(i) e_0(t,s) \equiv 1,\ e_p(t,t) \equiv 1;$

$(ii) e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s);$

$(iii) e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t);$

$(iv) e_p(t,s)e_p(s,r) = e_p(t,r);$

$(v) \dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s);$

$(vi) (e_p(t,s))^{\Delta} = p(t)e_p(t,s).$

Lemma 2 ([15]) Assume that $a > 0, b > 0$ and $-a \in \mathcal{R}^+$. Then

$$y^{\Delta}(t) \ge (\le) b - ay(t),\ y(t) > 0,\ t \in [t_0, +\infty)_{\mathbb{T}}$$

implies

$$y(t) \ge (\le) \frac{b}{a}[1 + (\frac{ay(t_0)}{b} - 1)e_{(-a)}(t,t_0)],\ t \in [t_0, +\infty)_{\mathbb{T}}.$$

Lemma 3 ([16]) Assume that $a > 0, b > 0, -b \in \mathcal{R}^+$, and $y(t) > 0, t \in [t_0, +\infty)_{\mathbb{T}}$.

(i) If $y^{\Delta}(t) \ge y(t)(b - ay(t))$, then $\liminf_{t\to+\infty} y(t) = \dfrac{b}{a}.$

(ii) If $y^{\Delta}(t) \le y(t)(b - ay(t))$, then $\limsup_{t\to+\infty} y(t) = \dfrac{b}{a}.$

Lemma 4 ([17])

Assume that $y(t) > 0, t \in \mathbb{T}$. Let $t \in \mathbb{T}^k$, if $y(t)$ is differentiable at t, then

$$\frac{y^{\Delta}(t)}{y(t)} \ge [\ln(y(t))]^{\Delta}.$$

Remark 1 If the time scale $\mathbb{T}$ is unbounded above, then $\mathbb{T}=\mathbb{T}^k$; therefore, in Lemma 4, if $\sup\mathbb{T}=+\infty$, then

$$\frac{y^{\Delta}(t)}{y(t)}\geq[\ln(y(t))]^{\Delta},\forall t\in\mathbb{T}.$$

Definition 1 ([18]) A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi=\{\tau\in\mathbb{R}:t\pm\tau\in\mathbb{T},\forall t\in\mathbb{T}\}\neq\{0\}.$$

Throughout this paper, we restrict our discussion on almost periodic time scales. Let $\mathbb{D}$ denotes $\mathbb{R}^n$ or an open subset of $\mathbb{R}^n$. The relevant definitions and the properties of almost periodic functions, see [18-20].

Definition 2 ([19]) $f\in C(\mathbb{T},\mathbb{R}^n)$ is an almost periodic function if and only if for any sequence $\alpha_n'\subset\Pi$, there exists a subsequence $\alpha_n\subset\alpha_n'$ such that $f(t+\alpha_n)$ converges uniformly on $\mathbb{T}$ as $n\to+\infty$. Furthermore, the limit function is also an almost periodic function.

Similar to the proof of Corollary 1.2 in [20], we can get the following lemma.

Lemma 5 Let $\mathbb{T}$ be an almost periodic time scale. If $f(t),g(t)$ are almost periodic functions, then, for any $\varepsilon>0, E\{\varepsilon,f\}\cap E\{\varepsilon,g\}$ is a nonempty relatively dense set in $\mathbb{T}$; that is, for any given $\varepsilon>0$, there exists a constant $l(\varepsilon)>0$, such that in any interval of length $l(\varepsilon)$, there exists at least a positive $\tau\in E\{\varepsilon,f\}\cap E\{\varepsilon,g\}$ such that

$$|f(t+\tau)-f(t)|<\varepsilon,|g(t+\tau)-g(t)|<\varepsilon,\forall t\in\mathbb{T}.$$

Consider the following almost periodic dynamic equation on time scale $\mathbb{T}$:

$$x^{\Delta}=f(t,x) \qquad (3)$$

and the associate product system of (3)

$$x^{\Delta}=f(t,x),\ y^{\Delta}=f(t,y). \qquad (4)$$

Lemma 6 ([19]) Suppose that there exists a Lyapunov function $V(t,x,y)\in C([0,+\infty)_{\mathbb{T}}\times\mathbb{D}\times\mathbb{D},\mathbb{R})$ satisfying the following conditions:

(1) $a(\|x-y\|)\leq V(t,x,y)\leq b(\|x-y\|)$, where $a,b\in K, K=\{a\in C(\mathbb{R}^+,\mathbb{R}^+):a(0)=0$ and a is increasing;

(2) $|V(t,x_1,y_1)-V(t,x_2,y_2)|\leq L(\|x_1-x_2\|+\|y_1-y_2\|)$, where $L>0$ is a constant;

(3) $D^+V^{\Delta}_{(3)}(t,x,y)\leq-\lambda V(t,x,y)$, where $-\lambda\in\mathcal{R}^+$ and $\lambda>0$.

Moreover, if there exists a solution $x(t)$ of (3) such that $x(t)\in S$, where $S\subset\mathbb{D}$ is a compact set. Then there exists a unique uniformly asymptotically stable almost periodic solution $p(t)$ of (3) in S. Furthermore, if $f(t,x)$ is periodic with period ω in t, then $p(t)$ is a periodic solution of (3) with period ω.

III. PERMANENCE

Assume that the coefficients of (1) satisfy

(H_1) $r_1^u-a_1^l>0;(r_2^u-a_2^l)+c_2^u(e^{M_{11}}-b_1)^2>0$;

(H_2) $\dfrac{r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^uM_{21}}{a_1^u}>1;\dfrac{r_2^l+c_2^l(e^{m_{11}}-b_1)^2-k_2^uM_{22}}{a_2^u}>1.$

Theorem 1 Let $(x_1(t),x_2(t),y_1(t),y_2(t))$ be any positive solution of system (1) with initial condition (2). If $(H_1)-(H_2)$ hold, then system (1) is permanent, that is, any positive solution $(x_1(t),x_2(t),y_1(t),y_2(t))$ of system (1) satisfies

$$m_{1i}\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq M_{1i}, \qquad (5)$$

$$m_{2i}\leq\liminf_{t\to+\infty}y_i(t)\leq\limsup_{t\to+\infty}y_i(t)\leq M_{2i}, \qquad (6)$$

especially if $m_{1i}\leq x_{i0}\leq M_{1i}, m_{2i}\leq y_{i0}\leq M_{2i}$, then

$$m_{1i}\leq x_i(t)\leq M_{1i},\ m_{2i}\leq y(t)\leq M_{2i},\ t\in[t_0,+\infty)_{\mathbb{T}},$$

where $i=1,2$, and

$$M_{11}=\frac{r_1^u-a_1^l}{a_1^l}, M_{12}=\frac{(r_2^u-a_2^l)+c_2^u(e^{M_{11}}-b_1)^2}{a_2^l}, M_{21}=\frac{d_1^u e^{M_{11}}}{h_1^l}, M_{22}=\frac{d_2^u e^{M_{12}}}{h_2^l},$$

$$m_{11}=\ln\Big(\frac{r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^u M_{21}}{a_1^u}\Big), m_{12}=\ln\Big(\frac{r_2^l+c_2^l(e^{m_{11}}-b_1)^2-k_2^u M_{22}}{a_2^u}\Big),$$

$$m_{21}=\frac{d_1^l e^{m_{11}}}{h_1^u}, m_{22}=\frac{d_2^l e^{m_{12}}}{h_2^u}.$$

Proof Assume that $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1) with initial condition (2). It follows from the first equation of system (1) and the inequality $e^x \geq 1+x$ for $x\in\mathbb{T}$ that

$$x_1^{\Delta}(t)\leq r_1(t)-a_1(t)(1+x_1(t))\leq (r_1^u-a_1^l)-a_1^l x_1(t). \tag{7}$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} x_1(t)\leq \frac{r_1^u-a_1^l}{a_1^l}\triangleq M_{11}.$$

Similarly, by Lemma 2, from the second equation of system (1), we can obtain

$$\limsup_{t\to+\infty} x_2(t)\leq \frac{(r_2^u-a_2^l)+c_2^u(e^{M_{11}}-b_1)^2}{a_2^l}\triangleq M_{12}.$$

Then, for arbitrarily small positive constant $\varepsilon>0$, there exists a $T_1>0$ such that

$$x_1(t)<M_{11}+\varepsilon, x_2(t)<M_{12}+\varepsilon, \forall t\in[T_1,+\infty]_{\mathbb{T}}.$$

From the third equation of system (1), when $t\in[T_1,+\infty)_{\mathbb{T}}$,

$$y_1^{\Delta}(t)<-h_1^l y_1(t)+d_1^u e^{M_{11}+\varepsilon}.$$

Let $\varepsilon\to 0$, then

$$y_1^{\Delta}(t)\leq -h_1^l y_1(t)+d_1^u e^{M_{11}}. \tag{8}$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} y_1(t)=\frac{d_1^u e^{M_{11}}}{h_1^l}\triangleq M_{21}.$$

Similarly, by Lemma 2, from the fourth equation of system (1), we can obtain

$$\limsup_{t\to+\infty} y_2(t)=\frac{d_2^u e^{M_{12}}}{h_2^l}\triangleq M_{22}.$$

Then, for arbitrarily small positive constant $\varepsilon>0$, there exists a $T_2>T_1$ such that

$$y_1(t)<M_{21}+\varepsilon, y_2(t)<M_{22}+\varepsilon, \forall t\in[T_2,+\infty]_{\mathbb{T}}.$$

On the other hand, from the first equation of system (1), when $t\in[T_2,+\infty)_{\mathbb{T}}$,

$$x_1^{\Delta}(t)>r_1^l-a_1^u\exp\{x_1(t)\}-c_1^u(e^{M_{12}+\varepsilon}-b_2)^2-k_1^u(M_{21}+\varepsilon).$$

Let $\varepsilon\to 0$, then

$$x_1^{\Delta}(t)\geq r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^u M_{21}-a_1^u\exp\{x_1(t)\}. \tag{9}$$

Let $z_1(t)=\exp\{x_1(t)\}$, then (9) can be written as

$$[\ln(z_1(t))]^{\Delta}\geq r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^u M_{21}-a_1^u z_1(t), \forall t\in[T_2,+\infty)_{\mathbb{T}}. \tag{10}$$

By Lemma 4 and Remark 1, it follows from (10) that

$$z_1^{\Delta}(t)\geq z_1(t)(r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^u M_{21}-a_1^u z_1(t)), \forall t\in[T_2,+\infty)_{\mathbb{T}}. \tag{11}$$

By Lemma 3, we can get

$$\liminf_{t\to+\infty} z_1(t)=\frac{r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^u M_{21}}{a_1^u},$$

that is

$$\liminf_{t\to+\infty} x_1(t)=\ln\Big(\frac{r_1^l-c_1^u(e^{M_{12}}-b_2)^2-k_1^u M_{21}}{a_1^u}\Big)\triangleq m_{11}.$$

Similarly, by Lemma 3, from the second equation of system (1), we can obtain

$$\liminf_{t\to+\infty} x_2(t)=\ln\Big(\frac{r_2^l+c_2^l(e^{m_{11}}-b_1)^2-k_2^u M_{22}}{a_2^u}\Big)\triangleq m_{12}.$$

Then, for arbitrarily small positive constant $\varepsilon>0$, there exists a $T_3 > T_2$ such that

$x_1(t) > m_{11} - \varepsilon, x_2(t) > m_{12} - \varepsilon, \forall t \in [T_3, +\infty]_{\mathbb{T}}$.

From the third equation of system (1), when $t \in [T_3, +\infty)_{\mathbb{T}}$,

$y_1^{\Delta}(t) > -h_1^u y_1(t) + d_1^l e^{m_{11}-\varepsilon}$.

Let $\varepsilon \to 0$, then

$$y_1^{\Delta}(t) \geq -h_1^u y_1(t) + d_1^l e^{m_{11}}. \quad (12)$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} y_1(t) = \frac{d_1^l e^{m_{11}}}{h_1^u} \triangleq m_{21}.$$

Similarly, by Lemma 2, from the fourth equation of system (1), we can obtain

$$\liminf_{t\to+\infty} y_2(t) = \frac{d_2^l e^{m_{12}}}{h_2^u} \triangleq m_{22}.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_4 > T_3$ such that

$y_1(t) > m_{21} - \varepsilon, y_2(t) > m_{22} - \varepsilon, \forall t \in [T_4, +\infty]_{\mathbb{T}}$.

In special case, if $m_{1i} \leq x_{i0} \leq M_{1i}, m_{2i} \leq y_{i0} \leq M_{2i}$, by Lemma 2 and Lemma 3, it follows from (7)-(8), (11)-(12) and the above analysis that

$m_{1i} \leq x_i(t) \leq M_{1i}, m_{2i} \leq y(t) \leq M_{2i}, t \in [t_0, +\infty)_{\mathbb{T}}$,

This completes the proof.

IV. ALMOST PERIODIC SOLUTION

Let $S(\mathbb{T})$ be the set of all solutions $(x_1(t), x_2(t), y_1(t), y_2(t))$ of system (1) satisfying

$m_{11} \leq x_1(t) \leq M_{11}, m_{12} \leq x_2(t) \leq M_{12}, m_{21} \leq y_1(t) \leq M_{21}, m_{22} \leq y_2(t) \leq M_{22}$

for all $t \in \mathbb{T}$.

Lemma 7 $S(\mathbb{T}) \neq \varnothing$.

Proof By Theorem 1, we see that for any $t_0 \in \mathbb{T}$ with

$m_{11} \leq x_{10} \leq M_{11}, m_{12} \leq x_{20} \leq M_{12}, m_{21} \leq y_{10} \leq M_{21}, m_{22} \leq y_{20} \leq M_{22}$,

system (1) has a solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ satisfying

$m_{11} \leq x_1(t) \leq M_{11}, m_{12} \leq x_2(t) \leq M_{12}, m_{21} \leq y_1(t) \leq M_{21}, m_{22} \leq y_2(t) \leq M_{22}, t \in [t_0, +\infty)_{\mathbb{T}}$.

Since $r_i(t), a_i(t), c_i(t), d_i(t), k_i(t), h_i(t), i = 1,2,$ are almost periodic, it follows from Lemma 5 that there exists a sequence $\{t_n\} \subset \Pi, t_n \to +\infty$ as $n \to +\infty$ such that

$r_i(t+t_n) \to r_i(t), a_i(t+t_n) \to a_i(t), c_i(t+t_n) \to c_i(t), d_i(t+t_n) \to d_i(t),$

$k_i(t+t_n) \to k_i(t), h_i(t+t_n) \to h_i(t), i = 1,2,$

as $n \to +\infty$ uniformly on $\mathbb{T}$.

Now, we claim that $\{x_i(t+t_n)\}, \{y_i(t+t_n)\}, i = 1,2,$ are uniformly bounded and equi-continuous on any bounded interval in $\mathbb{T}$. In fact, for any bounded interval $[\alpha, \beta]_{\mathbb{T}} \subset \mathbb{T}$, when n is large enough, $\alpha + t_n > t_0$, then $t + t_n > t_0, \forall t \in [\alpha, \beta]_{\mathbb{T}}$. So, $m_{11} \leq x_1(t+t_n) \leq M_{11}, m_{12} \leq x_2(t+t_n) \leq M_{12}, m_{21} \leq y_1(t+t_n) \leq M_{21}, m_{22} \leq y_2(t+t_n) \leq M_{22}$ for $t \in [\alpha, \beta]_{\mathbb{T}}$, that is, $\{x_i(t+t_n)\}, \{y_i(t+t_n)\}, i = 1,2,$ are uniformly bounded. On the other hand, $\forall t_1, t_2 \in [\alpha, \beta]_{\mathbb{T}}$, from the mean value theorem of differential calculus on time scales, we have

$$| x_1(t_1+t_n) - x_1(t_2+t_n) | \leq [r_1^u + a_1^u e^{M_{11}} + c_1^u (e^{M_{12}} - b_2)^2 + k_1^u M_{21}] | t_1 - t_2 |, \quad (13)$$

$$| x_2(t_1+t_n) - x_2(t_2+t_n) | \leq [r_2^u + a_2^u e^{M_{12}} + c_2^u (e^{M_{11}} - b_1)^2 + k_2^u M_{22}] | t_1 - t_2 |, \quad (14)$$

$$| y_1(t_1+t_n) - y_1(t_2+t_n) | \leq (h_1^u M_{21} + d_1^u e^{M_{11}}) | t_1 - t_2 |, \quad (15)$$

$$| y_2(t_1+t_n) - y_2(t_2+t_n) | \leq (h_2^u M_{22} + d_2^u e^{M_{12}}) | t_1 - t_2 |. \quad (16)$$

The inequalities (13)-(16) show that $\{x_i(t+t_n)\}, \{y_i(t+t_n)\}, i = 1,2,$ are equi-continuous on $[\alpha, \beta]_{\mathbb{T}}$. By the arbitrariness of $[\alpha, \beta]_{\mathbb{T}}$, the conclusion is valid.

By the Ascoli-Arzela theorem for time scales [21], there exists a subsequence of $\{t_n\}$, we still denote it as $\{t_n\}$, such that

$x_i(t+t_n) \to p_i(t), y_i(t+t_n) \to q_i(t), i=1,2,$

as $n \to +\infty$ uniformly in t on any bounded interval in $\mathbb{T}$.

Furthermore,

$x_1^{\Delta}(t+t_n) = r_1(t+t_n) - a_1(t+t_n)\exp\{x_1(t+t_n)\} - c_1(t+t_n)(\exp\{x_2(t+t_n)\} - b_2)^2 - k_1(t+t_n)y_1(t+t_n),$

$x_2^{\Delta}(t+t_n) = r_2(t+t_n) - a_2(t+t_n)\exp\{x_2(t+t_n)\} + c_2(t+t_n)(\exp\{x_1(t+t_n)\} - b_1)^2 - k_2(t+t_n)y_2(t+t_n),$

$y_1^{\Delta}(t+t_n) = -h_1(t+t_n)y_1(t+t_n) + d_1(t+t_n)\exp\{x_1(t+t_n)\},$

$y_2^{\Delta}(t+t_n) = -h_2(t+t_n)y_2(t+t_n) + d_2(t+t_n)\exp\{x_2(t+t_n)\}.$

Let $n \to +\infty$, then

$p_1^{\Delta}(t) = r_1(t) - a_1(t)\exp\{p_1(t)\} - c_1(t)(\exp\{p_2(t)\} - b_2)^2 - k_1(t)q_1(t),$

$p_2^{\Delta}(t) = r_2(t) - a_2(t)\exp\{p_2(t)\} + c_2(t)(\exp\{p_1(t)\} - b_1)^2 - k_2(t)q_2(t),$

$q_1^{\Delta}(t) = -h_1(t)q_1(t) + d_1(t)\exp\{p_1(t)\},$

$q_2^{\Delta}(t) = -h_2(t)q_2(t) + d_2(t)\exp\{p_2(t)\}.$

It is clear that $(p_1(t), p_2(t), q_1(t), q_2(t))$ is a solution of system (1). Moreover,

$m_{11} \le p_1(t) \le M_{11}, m_{12} \le p_2(t) \le M_{12}, m_{21} \le q_1(t) \le M_{21}, m_{22} \le q_2(t) \le M_{22}, \forall t \in \mathbb{T}.$

This completes the proof.

Remark 2 From the proofs of Theorem 1 and Lemma 7, we know that if the conditions of Theorem 1 hold, $S(\mathbb{T})$ is a positive invariant set of system (1).

Theorem 2 Suppose the conditions $(H_1)-(H_2)$ hold, assume further that

(H_3) $\lambda > 0$ and $-\lambda \in \mathcal{R}^+$, where

$\lambda = \min\{a_1^l - 2c_2^u(e^{M_{11}} - b_1) - d_1^u, a_2^l - 2c_1^u(e^{M12} - b_2) - d_2^u, h_1^l - k_1^u, h_2^l - k_2^u\};$

then there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of system (1) which is bounded on $S(\mathbb{T})$ for all $t \in \mathbb{T}$.

Proof From Lemma 7, system (1) has a bounded solution satisfying

$m_{11} \le x_1(t) \le M_{11}, m_{12} \le x_2(t) \le M_{12}, m_{21} \le y_1(t) \le M_{21}, m_{22} \le y_2(t) \le M_{22}, \forall t \in \mathbb{T},$

then

$|x_i(t)| < M_{1i}, |y_i(t)| < M_{2i}, i=1,2.$

For $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, we define the norm $\|(x_1, x_2, y_1, y_2)\| = |x_1| + |x_2| + |y_1| + |y_2|$.

Consider the product system of system (1)

$$
\begin{aligned}
&x_1^{\Delta}(t) = r_1(t) - a_1(t)\exp\{x_1(t)\} - c_1(t)(\exp\{x_2(t)\} - b_2)^2 - k_1(t)y_1(t),\\
&x_2^{\Delta}(t) = r_2(t) - a_2(t)\exp\{x_2(t)\} + c_2(t)(\exp\{x_1(t)\} - b_1)^2 - k_2(t)y_2(t),\\
&y_1^{\Delta}(t) = -h_1(t)y_1(t) + d_1(t)\exp\{x_1(t)\},\\
&y_2^{\Delta}(t) = -h_2(t)y_2(t) + d_2(t)\exp\{x_2(t)\},\\
&u_1^{\Delta}(t) = r_1(t) - a_1(t)\exp\{u_1(t)\} - c_1(t)(\exp\{u_2(t)\} - b_2)^2 - k_1(t)v_1(t),\\
&u_2^{\Delta}(t) = r_2(t) - a_2(t)\exp\{u_2(t)\} + c_2(t)(\exp\{u_1(t)\} - b_1)^2 - k_2(t)v_2(t),\\
&v_1^{\Delta}(t) = -h_1(t)v_1(t) + d_1(t)\exp\{u_1(t)\},\\
&v_2^{\Delta}(t) = -h_2(t)v_2(t) + d_2(t)\exp\{u_2(t)\},
\end{aligned}
\qquad (17)
$$

Suppose $X = (x_1(t), x_2(t), y_1(t), y_2(t)), Y = (u_1(t), u_2(t), v_1(t), v_2(t))$ are any two solutions of system (15), then $\|X\| \le B, \|Y\| \le B$, where $B = M_{11} + M_{12} + M_{21} + M_{22}$.

Consider a Lyapunov function defined on $\mathbb{T} \times S(\mathbb{T}) \times S(\mathbb{T})$ as follows

$$V(t, X, Y) = |x_1(t) - u_1(t)| + |x_2(t) - u_2(t)| + |y_1(t) - v_1(t)| + |y_2(t) - v_2(t)|. \qquad (18)$$

Since $\|X - Y\| = |x_1(t) - u_1(t)| + |x_2(t) - u_2(t)| + |y_1(t) - v_1(t)| + |y_2(t) - v_2(t)|$, we have

$$\frac{1}{2}\|X - Y\| \le V(t, X, Y) \le 2\|X - Y\|.$$

Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+), a(x) = \frac{1}{2}x, b(x) = 2x,$ thus the condition (1) in Lemma 6 is satisfied.

In addition,

$$
\begin{aligned}
&|V(t,X,Y)-V(t,\tilde{X},\tilde{Y})| \\
&=\big||x_1(t)-u_1(t)|+|x_2(t)-u_2(t)|+|y_1(t)-v_1(t)|+|y_2(t)-v_2(t)| \\
&-|\tilde{x}_1(t)-\tilde{u}_1(t)|-|\tilde{x}_2(t)-\tilde{u}_2(t)|-|\tilde{y}_1(t)-\tilde{v}_1(t)|-|\tilde{y}_2(t)-\tilde{v}_2(t)|\big| \\
&\le|x_1(t)-\tilde{x}_1(t)|+|x_2(t)-\tilde{x}_2(t)|+|y_1(t)-\tilde{y}_1(t)|+|y_2(t)-\tilde{y}_2(t)| \\
&+|u_1(t)-\tilde{u}_1(t)|+|u_2(t)-\tilde{u}_2(t)|+|v_1(t)-\tilde{v}_1(t)|+|v_2(t)-\tilde{v}_2(t)| \\
&=\|X-\tilde{X}\|+\|Y-\tilde{Y}\|.
\end{aligned}
$$

Let $L=1$, then the condition (2) of Lemma 6 is satisfied.

Finally, calculate the $V^{\Delta}(t,X,Y)$ along the solutions of (18), we can obtain

$$
\begin{aligned}
&V^{\Delta}(t,X,Y)=\mathrm{sgn}(x_1(t)-u_1(t))(x_1(t)-u_1(t))^{\Delta}+\mathrm{sgn}(x_2(t)-u_2(t))(x_2(t)-u_2(t))^{\Delta} \\
&+\mathrm{sgn}(y_1(t)-v_1(t))(y_1(t)-v_1(t))^{\Delta}+\mathrm{sgn}(y_2(t)-v_2(t))(y_2(t)-v_2(t))^{\Delta} \\
&=\mathrm{sgn}(x_1(t)-u_1(t))[-a_1(t)(\exp\{x_1(t)\}-\exp\{u_1(t)\})-c_1(t)((\exp\{x_2(t)\}-b_2)^2 \\
&-(\exp\{u_2(t)\}-b_2)^2)-k_1(t)(y_1(t)-v_1(t))] \\
&+\mathrm{sgn}(x_2(t)-u_2(t))r_2(t)[-a_2(t)(\exp\{x_2(t)\}-\exp\{u_2(t)\})+c_2(t)((\exp\{x_1(t)\}-b_1) \\
&-(\exp\{u_1(t)\}-b_1)^2)-k_2(t)(y_2(t)-v_2(t))] \\
&+\mathrm{sgn}(y_1(t)-v_1(t))[-h_1(t)(y_1(t)-v_1(t))+d_1(t)(\exp\{x_1(t)\}-\exp\{u_1(t)\})] \\
&+\mathrm{sgn}(y_2(t)-v_2(t))[-h_2(t)(y_2(t)-v_2(t))+d_2(t)(\exp\{x_2(t)\}-\exp\{u_2(t)\})].
\end{aligned} \tag{19}
$$

By using the mean value theorem, we have

$$\exp\{x_1(t)\}-\exp\{u_1(t)\}=\xi_1(t)(x_1(t)-u_1(t)),$$
$$\exp\{x_2(t)\}-\exp\{u_2(t)\}=\xi_2(t)(x_2(t)-u_2(t)),$$

where $\xi_1(t)$ lies between $\exp\{x_1(t)\}$ and $\exp\{u_1(t)\}$, and $\xi_2(t)$ lies between $\exp\{x_2(t)\}$ and $\exp\{u_2(t)\}$.

From (19), we have

$$
\begin{aligned}
&V^{\Delta}(t,X,Y)\le[-a_1(t)+c_2(t)(\exp\{x_1(t)\}+\exp\{u_1(t)\}-2b_1)+d_1(t)]\xi_1(t)|x_1(t)-u_1(t)| \\
&+[-a_2(t)+c_1(t)(\exp\{x_2(t)\}+\exp\{u_2(t)\}-2b_2)+d_2(t)]\xi_2(t)|x_2(t)-u_2(t)| \\
&+(-h_1(t)+k_1(t))|y_1(t)-v_1(t)|+(-h_2(t)+k_2(t))|y_2(t)-v_2(t)| \\
&\le-[a_1^l-2c_2^u(e^{M_{11}}-b_1)-d_1^u]e^{m_1}|x_1(t)-u_1(t)|-[a_2^l-2c_1^u(e^{M12}-b_2)-d_2^u]e^{m_2}|x_2(t)-u_2(t)| \\
&-(h_1^l-k_1^u)|y_1(t)-v_1(t)|-(h_2^l-k_2^u)|y_2(t)-v_2(t)|\le-\lambda V(t,X,Y),
\end{aligned}
$$

where $\lambda=\min\{a_1^l-2c_2^u(e^{M_{11}}-b_1)-d_1^u, a_2^l-2c_1^u(e^{M12}-b_2)-d_2^u, h_1^l-k_1^u, h_2^l-k_2^u\}$. From the condition (H_3), $\lambda>0$ and $-\lambda\in\mathcal{R}^+$, the condition (3) of Lemma 6 is satisfied.

To sum up, from Lemma 6, there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(t),x_2(t),y_1(t),y_2(t))$ of system (1) which is bounded on $S(\mathbb{T})$ for all $t\in\mathbb{T}$. This completes the proof.

Corollary 1 Assume that $(H_1)-(H_3)$ hold. Suppose that the nonnegative coefficients $r_i(t),a_i(t),c_i(t),d_i(t)$, $k_i(t)$, $h_i(t), i=1,2$ are periodic of period ω; then system (1) has a unique uniformly asymptotically stable periodic solution of period ω.

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