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Research Paper

The Approximate Solution of Non Linear Volterra Weakly Singular Integro-Differential Equations by Using Chebyshev Polynomials of the First Kind

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ABSTRACT :- *In this paper, we use Chebyshev polynomials method of the first kind of degree n to solve linear Fredholm weakly singular integro- differential equations (LFWSIDEs) of the second kind. This techniques transform the linear Fredholm weakly singular integro-differential equations to a system of a linear algebraic equations .Application are presented to illustrate the efficiency and accuracy of this method.*

Keywords: *Nonlinear Volterra integro-differential equation, Weakly singular kernel, Chebyshev polynomials, Trapezodial rule.*

I. INTRODUCTION

The introduction Integro - differential equations play an importance role in scientific field such as fluid dynamics, solid state physics , plasma physics and mathematical biology [1]. Some problems of mathematical physics are described in terms of (NLVWSIDEs) which has been studied by different methods including the spline collocation method [2], piecewise polynomials [3], Haar wavelets [4],the homotopy perturbation method (HPM)[5,6], the wavelet –Galerkind method [7] ,Taylor polynomials [8],The Tau method [9],the Sinc –collocation method [10],the Combined Laplace transform-Adomian decomposition method [11], and the Adomian's asymptotic decomposition method [12] to determine exact and approximate solutions. But to our knowledge there is still no viable analytic approach for solving weakly singular Volterra integro-differential equations. The present work is motivated by the desire to obtain approximate solution to nth-order nonlinear weakly singular Volterra integro-differential equation, where the integrand is weakly singular in the sense that its integral is continuous at the singular point. In section two definition of Chebyshev polynomials of the first kind and its properties with illustrative example are given. in section three the proposed method for solving nonlinear Volterra weakly singular integro-differential equations. Application is presented in section four. Finally, a brief Conclusion is stated in section five.

II. CHEBYSHEV POLYNOMIALS OF THE FIRST KIND $\mathbf{T_n(x)}$, [15]

The Chebyshev polynomials of the first kind of degree n is a set of orthogonal polynomials and it is defined by the recurrence relation

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \text{ for each } n \geq 1. \qquad \ldots(1)$$

2.1 Properties Of The Chebyshev Polynomials $T_n(x)$

1. The Chebyshev polynomials of the first kind $T_n(x), n = 0,1, \ldots$
   are a set of orthogonal polynomials over the interval [-1,1] with respect to the weight function $W(x) = (1 - x^2)^{-1/2}$, that is :

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \neq 0 \\ \pi & n = m = 0 \end{cases} \qquad \ldots(2)$$

2. The Chebyshev polynomials of the first kind can be defined by
   the trigonometric identity $T_n(\cos(\theta)) = \cos(n\theta)$ for n=0,1,2,3,… 3.
3. $T_n(x)$ has n distinct real roots x_i on the interval [-1,1], these roots are defined by :

$$x_i = \cos\left(\frac{(2i+1)\pi}{2N}\right), i = 0,1,2,\dots,N-1 \qquad \dots(3)$$

are called Chebyshev nodes. $T_n(x)$ assumes its absolute extrema at

$$x_j = \cos\left(\frac{j\pi}{N}\right) \text{ for } j = 0,1,2,\dots,N \qquad \dots(4)$$

4. A polynomial of degree N in Chebyshev form is a polynomial $p(x) = \sum_{n=0}^{N} a_n T_n(x)$ (5)
   Where T_n is the n^{th} Chebyshev form
   The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5for interval [-1,1]are given in figure(1).

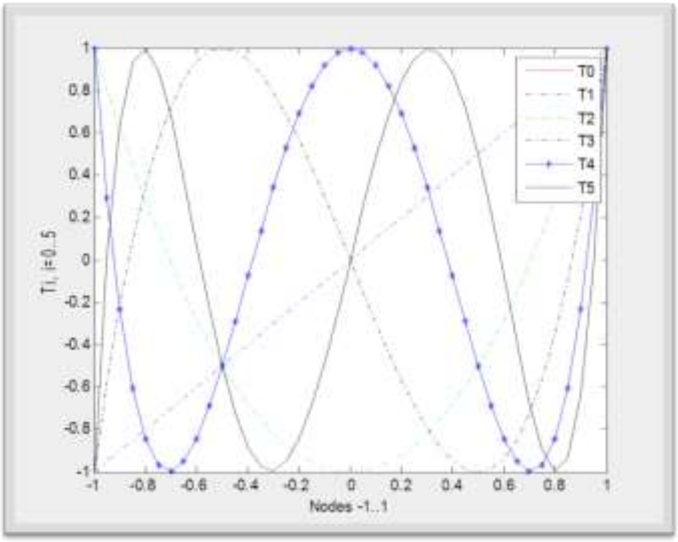


Figure (1): The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

2.2 Shifted Chebyshev Polynomials

Shifted Chebyshev polynomials are also of interest when the range of the independent variable is [0,1] instead of [-1,1]. The shifted Chebyshev polynomials of the first kind are defined as

$$T_n^*(x) = T_n(2x-1), \qquad 0 \le x \le 1 \qquad \dots(6)$$

Similarly, one can also build shifted polynomials for a generic interval [a,b] where

$$\overline{x_i} = \frac{b-a}{2} x_i + \frac{b+a}{2} \qquad \dots(7)$$

The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5 for interval [0,1] are given in figure(2).

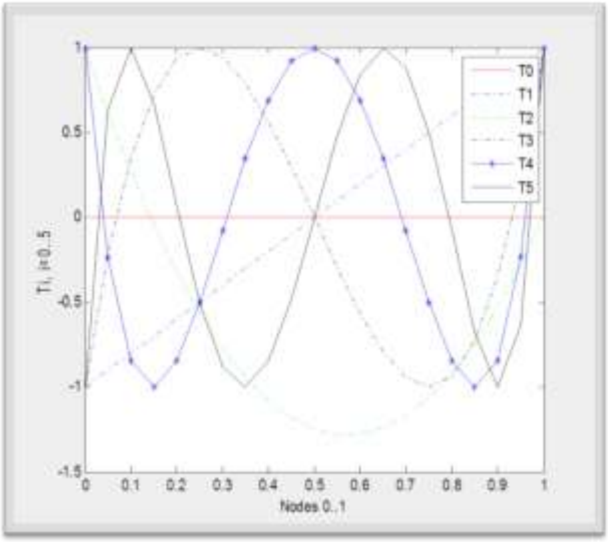


Figure (2):The first few Shifted Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

III. INDENTATIONS THE PROPOSED METHOD FOR SOLVING NONLINEAR VOLTERRA WEAKLY SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

Let us consider the following k^{th}-order NLVWSIDEs of the following form:

$$\sum_{j=0}^{k} p_j(x)y^{(j)}(x) = f(x) + \int_a^x k\big(x,t,y(t)\big)dt \quad x \in [a,b] \qquad \ldots(8)$$

with initial condition $y(a) = \beta$ where β is a constant

k , f are given functions and y is the solution to be determined. Moreover assume that the kernel $k\big(x,t,y(t)\big) = \frac{H(x,t,y(t))}{|x-t|^{\alpha}}$ $\forall x,t \in [a,b]$ with $x \neq t$ where $0 < \alpha < 1$. As well as assume that the kernel H is in $L^2[a,b]^2$ and the right hand side f are in $L^2[a,b]$. Also suppose that $H(x,t,y(t))$ satisfies in the Lipschitz condition,

$$|H\big(x_1,t,y(t)\big) - H(x_2,t,y(t))| \leq L_x|x_1 - x_2| \qquad \ldots(9)$$

An approximate solution of (9) where y(x) defined on [-1,1] is ,[14]:

$$y(\bar{x}) \cong \sum_{i=0}^{\infty} T_i(\bar{x})b_i \qquad \ldots(10)$$

If we truncated the series (10), then we can write (10) is written as follows:

$$y(\bar{x}) \cong \sum_{i=0}^{N} T_i(\bar{x})b_i \cong T(\bar{x})B \qquad \ldots(11)$$

$$k\big(\bar{x},t,y(t)\big) \cong k(\bar{x},t,\sum_{i=0}^{N} T_i(t)b_i) \cong k(\bar{x},t,T(t)B) \qquad \ldots(12)$$

$$\sum_{i=0}^{P} a_i(\bar{x})y^{(i)}(\bar{x}) \cong \sum_{i=0}^{p} a_i(\bar{x})\frac{d^{(i)}}{dx^i}\sum_{i=0}^{n} T_i(\bar{x})b_i \qquad \ldots(13)$$

where $T(\bar{x}) = [T_0(\bar{x}), T_1(\bar{x}), T_2(\bar{x}), \ldots, T_N(\bar{x})]$,
$B = [b_0, b_1, b_2, \ldots, b_N]^T$

clearly T is $1 \times (N+1)$ vectors and B is $(N+1) \times 1$ vectors .then the aim is to find Chebyshev coefficients , that is the matrix B. We first substitute the Chebyshev nodes, which are defined by

$$\bar{x}_i = \cos\left(\frac{(2i+1)\pi}{2N}\right), i = 0,1,2,\ldots,N-1$$

into (12) and (13) and (14) then rearrange a new matrix form to determine B:

$$y = f + k \qquad \ldots(14)$$

In which k is the nonlinear integral part of (8) and

$$y = \begin{pmatrix} p_0(\bar{x}_0)y^{(0)}(\bar{x}_0) \\ p_1(\bar{x}_1)y^{(1)}(\bar{x}_1) \\ . \\ . \\ . \\ p_k(\bar{x}_N)y^{(k)}(\bar{x}_N) \end{pmatrix}, f = \begin{pmatrix} f(\bar{x}_0) \\ f(\bar{x}_1) \\ . \\ . \\ . \\ f(\bar{x}_N) \end{pmatrix}, \quad k = \begin{pmatrix} k(\bar{x}_0) \\ k(\bar{x}_1) \\ . \\ . \\ . \\ k(\bar{x}_N) \end{pmatrix} \qquad \ldots(15)$$

By substituting (11) , (12) and (13) into (14) gives nonlinear algebraic equations in (N+1) unknown coefficients .These equations are solved by using (Matlab R2010b) to obtain the unknown coefficients B which are then substitute into (10) to get the approximate solution of (8). If the function $y(x)$ defined on $[0,1]$. the transformation $\tilde{x} = \frac{1}{2}[(b-a)\bar{x} + (a+b)]$ is used to transform the nodes $\bar{x}_i$ on $[-1,1]$ into the corresponding nodes $\tilde{x}_i$ on $[0,1]$.

Two algorithms for solving (NLVWSIDE's) on interval [-1,1] and [0,1] are given as follows:

3.1 The Algorithm For Solving (Nlvwsides) In To [-1,1]

Input : $a, b, \alpha, N, M, m, y(x), f(x), p(x), \square$.
Output : The approximate solution of the NLVWSIDEs.
Step 1: process: Find $T_i(\bar{x}), (T_i(\bar{x}))'$. (Chebyshev polynomials)
Step 2: Find roots $\bar{x}_i, i = 0, \ldots, N$. (roots of Chebyshev polynomials)
Step 3: Find roots
$t_{ij} = a + j + k_i \pm \square, k_i = \frac{\bar{x}_i - a}{M}, i = 0, \ldots, N, j = 0, \ldots, M, \bar{x} = t.$
Step 4: Calculate $(T_{ij}(\bar{x}))^m$ where $m > 1$ $i, j = 0,1, \ldots, N$
Step 5: Calculate $R_i = \frac{k_i}{2}[Y_{i0} + 2\sum_{k=1}^{M-1} Y_{ik} + Y_{iM}]$ (Trapezoidal rule)
Where $Y_{ij} = \frac{(T_{ij}(t))^m}{|\bar{x}_i - t_{ij}|^{\alpha}}$, $i = 0, \ldots, N, j = 0, \ldots, M$
Step 6: Construct the system:
$A_{ik} = b_i * \left(\left(T_k(\bar{x}_i)\right)' + p(\bar{x}_i) * T_k(\bar{x}_i)\right) - (b_i)^m * R_i, B_i = f(\bar{x}_i)$
$i, k = 0, \ldots, N$ where b_i are unknowns.
Step 7: let $b_i = 0, i = 0, \ldots, N$. (initial values)
Step 8: Solve the non-linear system $A_{ik} = B_i$ using the library function f

solve and find the unknowns b_i.
Step 9: Calculate the approximate function $y_N(\bar{x}) = \sum_{i=0}^{N} T_i(\bar{x}) b_i$
Step 10: Calculate absolute error is the comparison between the exact and the approximate solutions.
Step 11: END of the process.

3.2 The Algorithm For Solving (Nlvwsides) In To[0,1]

Input: $a, b, \alpha, N, M, m, y(x), f(x), p(x), \square$.
Output: The approximate solution of the NLVWSIDEs.
Step 1 : process : Find $T_i(\bar{x})$. (Chebyshev polynomials)
Step 2 : Find $T_i^*(\tilde{x}), (T_i^*(\tilde{x}))'$. (shifted Chebyshev polynomials)
Step 3 : Find roots $\bar{x}_i, i = 0, \dots, N$ (roots of Chebyshev polynomials)
Step 4: Find roots $\tilde{x}_i$ by using the transformation
$\tilde{x}_i = \frac{1}{2}[(b-a)\bar{x}_i + (a+b)], i = 0,1,\dots,N$ (roots of shifted Chebyshev polynomials)
Step 5: Calculate $(T_{ij}^*(\tilde{x}))^m$ where $m > 1$ $i, j = 0,1,\dots,N$
Step 6: Find roots
$\tilde{t}_{ij}^{\,*} = a + j + k_i \pm \square, k_i = \frac{\tilde{x}_i - a}{M}, i = 0,\dots,N, j = 0,\dots,M, \tilde{x} = t.$
Step 7: Calculate $R_i = \frac{k_i}{2}[Y_{i0} + 2\sum_{k=1}^{M-1} Y_{ik} + Y_{iM}]$ (Trapezoidal rule)
Where $Y_{ij} = \frac{(T_{ij}^*(\tilde{t}))^m}{|\tilde{x}_i - \tilde{t}_{ij}^{\,*}|^{\alpha}}, i = 0,\dots,N, j = 0,\dots,M$
Step 8: Construct the system:
$A_{ik} = b_i * ((T_k^*(\tilde{x}_i))' + p(\tilde{x}_i) * T_k^*(\tilde{x}_i)) - (b_i)^m * R_i, B_i = f(\tilde{x}_i) i, k = 0,\dots,N$
where b_i are unknowns.
Step 9: let $b_i = 0, i = 0,\dots,N$ (initial values)
Step 10: Solve the non-linear system $A_{ik} = B_i$ using the library function f solve and find the unknowns b_i.
Step 11 : Calculate the approximate function $y_N(\tilde{x}) = \sum_{i=0}^{N} T_i^*(\tilde{x}) b_i$
Step 12: Calculate absolute error is the comparison between the exact and the approximate solutions.
Step 13: END of the process.

I. APPLICATION

In this section, A numerical example to clarify the applicability and activity of the proposed method is given. The computations have been performed by using Matlab R2010b.

Consider the following first-order NLVWSIDEs,[13]:

$$y'(\tilde{x}) + p(\tilde{x})y(\tilde{x}) = f(\tilde{x}) + \int_a^{\tilde{x}} \frac{F(y(\tilde{t}))}{(\tilde{x}-\tilde{t})^{\alpha}} dt, 0 < \alpha < 1, \tilde{x} \in [0,1] \qquad \dots(16)$$

with initial condition $y(0) = 0$

where $\alpha = \frac{1}{2}, a = 0$, and the exact solution $y(\tilde{x}) = \tilde{x}(\tilde{x}-1)$. Let $F(y(\tilde{t})) = y^2(\tilde{t})$,
$p(\tilde{x}) = \left(\frac{16}{315}\right)\tilde{x}^{\frac{5}{2}}(21 + 4\tilde{x}(4\tilde{x}-9)) + 1, f(\tilde{x}) = \tilde{x}^2 + \tilde{x} - 1.$

Tables (1), (2) and (3) illustrate the comparison between the exact and the approximate solution depending on Mean Square Error(MSE) and Elapsed Time(ET) for N=10,12 and 14 respectively

Table (1): Results obtained and errors for application: N=10

$\tilde{x}$-value	**Exact solution**	**Approximate solution**	**Absolute error**
0.0051	-0.0051	-0.0234	0.0183
0.0452	-0.0431	-0.0593	0.0162
0.1221	-0.1072	-0.1193	0.0121
0.2297	-0.1769	-0.1828	0.0059
0.3591	-0.2302	-0.2268	0.0034
0.5	-0.2500	-0.2344	0.0156
0.6409	-0.2302	-0.2018	0.0283
0.7703	-0.1769	-0.1387	0.0382
0.8779	-0.1072	-0.0631	0.0441
0.9548	-0.0431	0.0038	0.0470
0.9949	-0.0051	0.0430	0.0481
MSE	**8.8719e-004**	**ET**	**164.313091Sec.**

Table (2): Results obtained and errors for application: N=12

$\tilde{x}$-value	**Exact solution**	**Approximate solution**	**Absolute error**

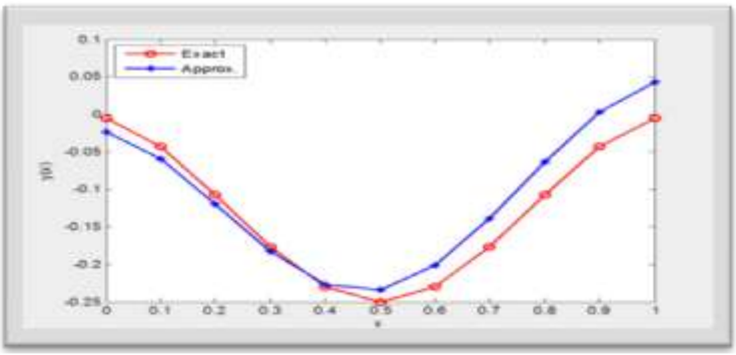
0.0037	-0.0036	-0.0164	0.0127
0.0325	-0.0314	-0.0429	0.0114
0.0885	-0.0807	-0.0896	0.0090
0.1684	-0.1401	-0.1453	0.0053
0.2676	-0.1960	-0.1959	0.0001
0.3803	-0.2357	-0.2279	0.0078
0.5	-0.2500	-0.2325	0.0175
0.6197	-0.2357	-0.2080	0.0277
0.7324	-0.1960	-0.1598	0.0362
0.8316	-0.1401	-0.0981	0.0420
0.9115	-0.0807	-0.0354	0.0453
0.9675	-0.0314	0.0155	0.0470
0.9964	-0.0036	0.0440	0.0477
MSE	**8.5709e-004**	**ET**	**420.942616Sec.**

Table (3): Results obtained and errors for application: N=14

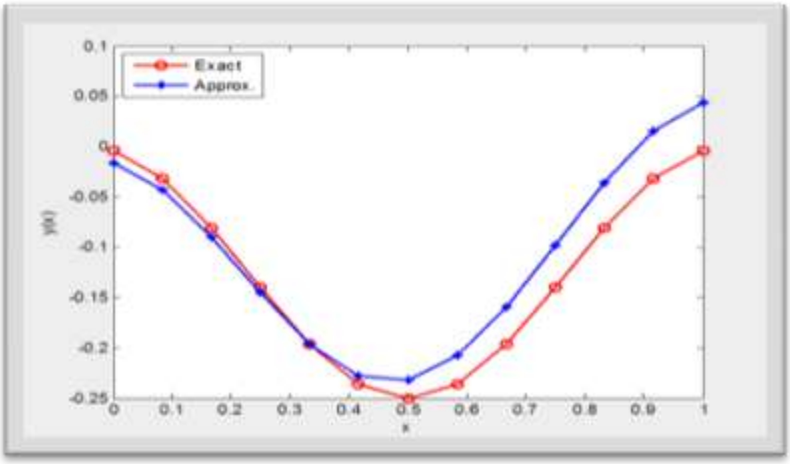
x̃-value	**Exact solution**	**Approximate solution**	**Absolute error**
0.0027	-0.0027	-0.0120	0.0093
0.0245	-0.0239	-0.0323	0.0084
0.067	-0.0625	-0.0693	0.0068
0.1284	-0.1119	-0.1163	0.0043
0.2061	-0.1636	-0.1645	0.0009
0.2966	-0.2086	-0.2047	0.0039
0.396	-0.2392	-0.2287	0.0105
0.5	-0.2500	-0.2314	0.0186
0.604	-0.2392	-0.2121	0.0271
0.7034	-0.2086	-0.1741	0.0346
0.7939	-0.1636	-0.1234	0.0402
0.8716	-0.1119	-0.0682	0.0438
0.933	-0.0625	-0.0167	0.0458
0.9755	-0.0239	0.0230	0.0469
0.9973	-0.0027	0.0447	0.0474
MSE	**8.4706e-004**	**ET**	**5641.463816Sec.**

Figures (3),(4) and (5) illustrate the comparison between the exact and the approximate solution for N=10,12 and 14 respectively

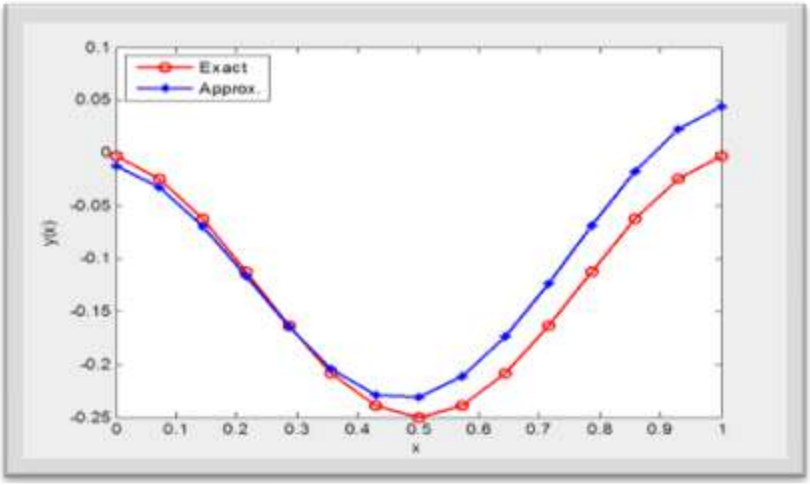
Figure(6) illustrate the comparison between the shifted Chebyshev nodes and the MSE of application when N=2…14.



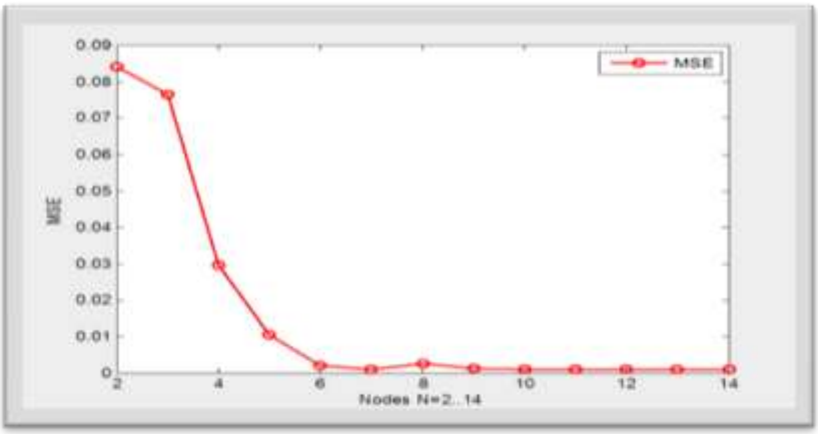
Figure(3): A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of application for N=10.



Figure(4):A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of application for N=12.



Figure(5):A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of application for N=14.



Figure(6):A Comparison between the Shifted Chebyshev nodes and the MSE of application When N=2….14.

III. CONCLUSION

In this paper, the expansion method is using Chebyshev polynomials of the first kind of degree n as basis function for approximating the solution of one weakly singular integro-differential equations: which are the NLVWSIDEs. In application the solution of NLVWSIDEs is reduced to the system of nonlinear equations by removing the singularity using an approximate point t, and we have the following results:

- When the degree of expansion method of Chebyshev polynomials of the first kind is increases the error is decreases. Which is shown in Tables (1), (2), and(3).
- The proposed method is a delicate and an effective to solve NLVWSIDEs.
- This method can be extended and applied to the system of non linear Volterra weakly singular integro-differential equations.

REFERENCES

[1]. Belbas. S. A (2007), A new method for optimal control of Volterra integral equations, *Journal of mathematics and computation* 189, 2007, pp1902-1915.

[2]. H. Beunner, On the numerical solution of nonlinear Volterra –Fredholm integral equations by collocation methods, *SIAM Journal on numerical analysis* ,Vol.27,no.4, 2008, pp.495-508.

[3]. D.Conte and I. D. Prete, Fast collocation methods for Volterra integral equations of convolution type, *Journal of computational and applied mathematics*,vol.196,no.2, 2006, pp.652-663.

[4]. O. Lepik, Haar Wavelet method for nonlinear integro-differential equations, *Applied mathematics and computation* ,vol.176,no.1,2006, pp.324-333.

[5]. M. Ghasemi, M. TavassoliKajani, and E. Babolian, Application of He's homotopy perturbation method to nonlinear integro-differential equations", *Applied mathematics and computation*,Vol.188, no.1, 2007, pp.538-548.

[6]. J.Saberi-Nadjafi and A.Ghorbani, He's homotopy perturbation method : an effective tool for solving nonlinear integral and integro-differential equations, *Computers and mathematics with applications*,Vol.58,no.11-12, 2009, pp.2379-2390.

[7]. Y. Mahmoudi, Wavelet Galerkin method for numerical solution of nonlinear integral equation, *Applied mathematics and computation*, Vol.167,no.2,2005, pp.1119-1129.

[8]. K.Maleknejad and Y.Mahmoudi, Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations, *Applied mathematics and computation*, Vol.145, no.2-3, 2003, pp.641-653.

[9]. G. Ebadi, M. Y. Rahimi-Ardabili, and Shahmorad, Numerical solution of the nonlinear Volterra integro-differential equations by the tau method, *Applied mathematics and computation*,Vol.188,no.2, 2007, pp.1580-1586.

[10]. M. Zarebnia and Z.Nikpour (2010):,"Solution of linear Volterra integro-differential equations via Sinc functions", International Journal of Applied Mathematics and Computer,Vol.2,pp.1-10.

[11]. A. M.Wazwaz, The combined Laplace transform-Adomain decomposition method for handling nonlinear Volterra integro-differential equations, *Applied mathematics and computation*,Vol.216, no.4, 2010, pp.1304-1309.

[12]. L. Bougoffa, R. C. Rach, and A. Mennouni, An approximate method for solving a class of weakly singular Volterra integro-differential equations, *Applied mathematics and computation*,Vol.217, no.22, 2011, pp.8907-8913.

[13]. Xueqin Lv and Sixing Shi, The Combined RKM and ADM for solving Nonlinear weakly singular Volterra integro-differential equations, volume 2012, Article ID 258067,10pages.

[14]. Qinghua Wu, The Approximate solution of fredholm integral equtions with oscillatory trigonometric kernels", volume 2014, Article ID 172327,7pages.

[15]. J. C. Mason, D. C. Handscomb, *Chebyshev polynomials*, (Boca Raton London New York Washington, 2003).