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Research Paper

A mathematical study and numerical simulation of a fluid particle inside a vortex

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Abstract:
In this research paper, we proposed to study the behavior of a fluid particle inside a vortex with time-independent or time-dependent of speed components. This study is realized in three dimension. In preliminaries, we have defined some properties and calculate some of the terms necessary for our study. We have shown in the case where speed is independent of time, an impossibility of having an incompressibility situation in flow and that the value of the density does not affect the pressure of the particle. We have also shown in the case where the speed is time dependent, the possibility of having an incompression situation if the partial derivative equations or the value of t check certain conditions, and that in this case the density has strong influence on the pressure of the particle which depends on the position. The simulation in the case where v is independent on time and the external forces are not functions of x, y, z shows that the pressure follows a increasing linear behavior for a power from $n = 2$ to $n = 5$, a resemblance of pressure according to whether that n is even or odd. In the case that v depends on time, the velocity remains different and does not show the sinusoidal but a rectilinear pace. The four speeds appear to be stackable but it is not between the different models. We show that the trajectory followed by the particle influences the pressure, a small resemblance of the pressure between the models two by two. The simulation of ratatinal, divergence and pressure together show that the trajectory of the fluid particle has an influence on these variables. And a small resemblance have been seen between the models two by two.

Keywords: *Deformation kinematics, gradient tensor of speed, incompressibility, rotational, divergence, theorem of Bernoulli, viscosity, equations of navier-stokes.*

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I. Introduction

The notion of fluid refers to the absence of an organized structure of matter at the microscopic level, which allows a large capacity for movement of the atoms that constitute a fluid. It is for this reason that the fluid state is represented by the liquid and gaseous bodies [1]. The study of fluids plays a very important role nowadays because it allows the development of models which make it possible to improve the performance of 1 machines in the maritime, terrestrial and space fields. A particle of a fluid can be viscous, compressible, or incompressible. When we are interested in viscosity, an exact analysis of the radiative effect of the free magnetohydrodynamics (MHD) of convection flow of an incompressible viscous fluid on a vertical plate is studied and the continuity, the moment and the energy equations are solved using an appropriate transformation [2]. An explicit form solution of the moment diffusion equation for a viscous fluid flowing around a plateau taking into account the deceleration with three regions characteristic of the viscous flow at is given in [3]. In the case of an incompressible flow, a new method of scalar projection presented for the simulation of incompressible flow with a variable density is proposed with a first phase of purely kinematic projection. The predicted speed is subjected to a discrete Hodge-Helmholtz decomposition, [4]. A solution of an incompressible fluid flow is also studied in [5]. However the study of the kinematics of fluid flow plays a very important role by what it constitutes in most of the time, the starting point of a study in mechanics of continuous mediums in [6], [7] and [8]. Incompressibility, divergence and rotational have been studied more recently in new kinematics of vortex flow in [9]. The further away the particle is from the center, the more volume and speed). In this document, we first start with a preliminary study of the mechanics of fluid flow by defining some tensor used in this study. We are going to apply some tensors defined in the preliminaries to two examples of vortex flow with time dependent or non-time dependent velocity components, in order to determine the pressure, rotational and divergence of these kinematics. As a contribution, we will shown that in a flow where the velocity components are independent of time, the condition of incompressibility is impossible and that the volume density has no influence on the pressure. We will also show that in the case of a dependence of the speed on the time, the value of the volume density influences the pressure which also depend on the position of the particle and that with the partial derivatives checking certain conditions or for a specific value of t, we can have a situation of incompressibility in this case of fluid flow. Finally the simulation and the results will allow us to see the correlation of our study compared to the reality, and this is what allowed us to validate our study.

II. Preliminaries

In mechanics of continuous mediums as in the particular case of fluid mechanics, a kinematics of transformation is always given from one of the two following configurations that are: the Lagrangian configuration and the Eulerian configuration. The Lagrangian configuration observe the fluid particle passed in a fixed point of the space while the Eulerian configuration follows the particle continuous mediums in its movement.

$$x_i = f_i(X); \tag{1}$$

where $f_{i(1\leq i\leq 3)}$ are one to one applications and $X = (X_1, X_2, X_3)$ in the case of a spatial transformation.
It is possible that a kinematic of transformation becomes defined from the velocity components as:

$$v_i = g_i(X); \tag{2}$$

To better understand the behavior of a fluid, we define the speed tensor noted $\mathbf{D}$ and which is defined by:

$$\mathbf{D} = D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right); \tag{3}$$

From the speed tensor, we can define the stress tensor noted σ which is a function of the first.

$$\sigma = h\left(\mathbf{D}\right); \tag{4}$$

An isotropic medium is a medium whose properties are identical regardless of the direction of observation. Fluids are generally considered to be isotropic medium [A].
we can calculate the three isotropic elementary invariants which are:

$$\begin{aligned} I_1 &= tr(\mathbf{D}); \\ I_2 &= tr(\mathbf{D}^*); \\ I_3 &= det(\mathbf{D}); \end{aligned} \tag{5}$$

where $\mathbf{D}^*$ is the adjoint tensors of $\mathbf{D}$, tr is the $trace$ operator and det is the $determinant$ operator.
The incompressibility conditions are given by:

$$\begin{aligned} I_1 &= 0; \\ I_3 &= 1. \end{aligned} \tag{6}$$

Let us now consider a fluid particle of volume V with an external pressure P_e. This fluid will be said to be incompressible if its vulume varies when the external pressure varies, Mathematically, that means that:

$$\frac{\partial P_e}{\partial t} \neq 0 \Longrightarrow \frac{\partial V}{\partial t} \neq 0 \tag{7}$$

Compressibility is a characteristic property of gases. Most liquids such as petroleum are incompressible. It is from the same that the principle of Archimed comes to us.
Let Q be a regular bounded domain of $\mathbb{R}^n$, ∂Q the border of Q and $u \in C^1\left(\overline{Q}\right)$ a continuous vector field. So the Ostrograsdski's theorem gives us:

$$\int_{\partial Q} u.\eta ds = \int_Q div\left(u\right) du; \tag{8}$$

with η the unit normal oriented outward.
It should be noted that a simply related field can be regular under certain conditions. If Ω is a connected regular bounded domain of connected components ϕ, then the connected components restrictions are paramerizations absolutely continues
The general or integral formula for the conservation of mass at time t also applies to a fluid, which explains the use of the integral tool. So the mass of a fluid can be written as:

$$m\left(t\right) = \int_Q du_t\left(x\right) \tag{9}$$

The density of a substance noted ρ, also called density of mass, is a physical quantity which characterizes the mass of this substance per unit of volume. In the case where $du_t\left(x\right) = \rho\left(x, t\right)$, we obtain:

$$m\left(t\right) = \int_Q \rho\left(x, t\right) \tag{10}$$

The conservation of mass is a fundamental law of the mechanics of continuous mediums. It indicates that during the whole transformation, the mass is conserved :

$$\frac{d}{dt} m\left(t\right) = \frac{d}{dt} \int_Q \rho\left(x, t\right) = 0, \forall Q\left(t\right). \tag{11}$$

The mechanical deformations of continuous mediums are one-to-one transformations between the initial configuration and the deformed configuration, which gives us:

$$x_i = f_i\left(X\right) \Longleftrightarrow X_i = \phi_i\left(x\right), \tag{12}$$

where $\phi_{i(1\leq i\leq 3)}$ are the reverse of $f_{i(1\leq i\leq 3)}$

The Lagrange representation makes it possible not only to follow a particle in its movement but also to study the particle derivative or total derivative or Lagrangian derivative which takes into account not only the local variation of the parameter over time but also the variation of that - here related to the displacement of the particle.
We define the particle derivative by:

$$\begin{aligned}&\frac{d}{dt}f(X,t)=\frac{\partial}{\partial t}f(X,t)+\\&\frac{\partial}{\partial x}f(\phi(x),t)\frac{\partial}{\partial x}\phi(x,t),\end{aligned} \tag{13}$$

What finally yields:

$$\frac{d}{dt}f=\frac{\partial}{\partial t}f+(\nabla f).v, \tag{14}$$

The term convective or convection term represented by $(\nabla f).v$ refers to the heat transfers occurring between a surface and a moving fluid when these are at different temperatures. In addition to the energy transfer due to diffusion, there is also transfer through the movement of the fluid.

Using the density function in the previous relation withe the condition of (10), we get:

$$\int_{Q_t}\left[\frac{\partial}{\partial t}\rho(x,t)+\rho(x,t)\,div\left(\vec{v}\right)\right]dx=0, \tag{15}$$

and the integral on a segment of a continuous function of constant sign such as the density is zero if and only if this function is zero. So

$$\frac{\partial}{\partial t}\rho(x,t)+\rho(x,t)\,div\,(\vec{v})=0, \tag{16}$$

$\vec{v}$ is the velocity of the fluid particle.
This previous relation defines an important principle called the continuity equation which describes the principle of conservation of mass in several different forms: local conservative (derivative in normal time), non local-conservative (the derivative in time follows the particle in its movement), or integral.

In differential calculus, the Reynolds transport theorem (also known as the Leibniz-Reynolds transport theorem), or in short Reynolds theorem, is a three-dimensional generalization of the Leibniz integral rule which is also known as differentiation under the integral sign.
This theorem is stated as follows:
Let Q be a regular domain of $\mathbb{R}^n$, ∂Q the frontier of Q and v_n the outgoing normal speed at point $x\in\partial Q$. Then for any tensor field continu admitting a temporal derivative, we have:

$$\frac{d}{dt}\int_Q \rho dv = \int_Q \partial_t \rho dv + \int_{\partial Q} \rho\,(v.\boldsymbol{\eta})\,ds, \tag{17}$$

with $\boldsymbol{\eta}$ the unit normal oriented outward.
You should know that this theorem is fundamental because it makes it possible to obtain all the fundamentals relations of mechanics.

The law of conservation of momentum states that the total momentum of a system before a transformation is equal to the total momentum in the same system after the transformation.
When we replace ρ by ρv in the Leibniz-Reynolds transport theorem, we obtain:

$$\frac{d}{dt}\int_Q \rho v dv = \int_Q \partial_t \rho v dv + \int_{\partial Q} \rho v\,(v.\boldsymbol{\eta})\,ds, \tag{18}$$

Accordind to the Ostragrasdski theorem, we have:

$$\int_{\partial Q} \rho v\,(v.\boldsymbol{\eta})\,ds = \int_Q div\,(\rho v \otimes v)\,dv \tag{19}$$

By using (18) in (17) and removing the term from the integral, we end up with:

$$\frac{d}{dt}\rho v = \partial_t \rho v + div\,(\rho v \otimes v)\,, \tag{20}$$

The fundamental principle of dynamics is that any variation in the amount of motion results from the application of forces. So the general relation of conservation of the momentum, the divergence theorem and the momentum conservative law give:

$$\partial_t\,(\rho \mathbf{v}) + div\,(\rho \mathbf{v} \otimes \mathbf{v}) = div\,(\boldsymbol{\sigma}) + f, \tag{21}$$

$\boldsymbol{\sigma} = -p\mathbf{I} + 2\gamma\mathbf{D} + \lambda div\,(\mathbf{u})$ is the Cauchy stress tensor which allows to characterize the state of stress, that is to say the internal forces brought into play between the deformed portions of a material in mechanics of continuous mediums, and $\mathbf{I}$ the identity tensor.

The criterion of incomprehensibility is also defined by the relation:

$$div\,(v) = 0. \tag{22}$$

as we pointed out at the beginning defines incompressibility as a non-variation of the volume as a function of time, this will also imply a non-variation of the density. So we have:

$$\rho\,(x,t) = \rho_0. \tag{23}$$

We have several mathematical formulations to study the movement of a fluid. In fluid mechanics, the Navier-Stokes equations are nonlinear partial differential equations that describe the motion of Newtonian fluids.
In the case of a spatial and in incompressible, these equations are defined by:

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$$\rho_0 \frac{d}{dt} v_i + \frac{\partial p}{\partial x_i} = 2\mu \Delta v_i + f_i, i = 1, 2, 3. \tag{24}$$

A fluid is defined Newtonian if the behavior model which gives the Cauchy stress tensor $\boldsymbol{\sigma}$ as a function of d is an affine relation, i.e.

$$\boldsymbol{\sigma} = -p\mathbf{I} + \lambda D_{ll}\mathbf{I} + 2\mu\mathbf{D} \tag{25}$$

with p, λ and μ independent scalars of $\mathbf{D}$.
Viscosity can be defined as the set of phenomena of resistance to the movement of a fluid for a flow with or without turbulence. The high viscosity decreases the freedom of flow of the fluid and dissipates its energy.

3 Application to a vortex flow

Let now consider a particle on an domain Ω of $\mathbb{R}^3$ with the following flow kinetics defined as:

$$\begin{aligned} x_x = v_0 g_1(X,t); \quad x_y &= v_0 g_2(X,t); \\ x_z &= v_0 g_3(X,t); \end{aligned} \tag{26}$$

With our flow kinematics, the velocity components are described by the following kinematics:

$$\begin{aligned} v_x = v_0 \dot{g}_1(X,t); \quad v_y &= v_0 \dot{g}_2(X,t); \\ v_z &= v_0 \dot{g}_3(X,t); \end{aligned} \tag{27}$$

with v_0 the initial velocity of the fluide paticle and g_i functions of $x = (x, y, z)$ verifying :
$\dot{g}_1(X,0) = \dot{g}_2(X,0) = \dot{g}_3(X,0) = 1.$
The components of the gradient tensor of velocity become:

$$\begin{aligned} D_{11} = v_0 \frac{\partial^2}{\partial x^2}\dot{g}_1; \quad D_{22} &= v_0 \frac{\partial^2}{\partial y^2}\dot{g}_2; \\ D_{33} &= v_0 \frac{\partial^2}{\partial z^2}\dot{g}_3 \\ D_{12} = D_{21} &= \frac{1}{2}\left(\frac{\partial}{\partial y}\dot{g}_1 + \frac{\partial}{\partial x}\dot{g}_2\right); \\ D_{13} = D_{31} &= \frac{1}{2}\left(\frac{\partial}{\partial z}\dot{g}_1 + \frac{\partial}{\partial x}\dot{g}_3\right); \\ D_{23} = D_{32} &= \frac{1}{2}\left(\frac{\partial}{\partial z}\dot{g}_2 + \frac{\partial}{\partial y}\dot{g}_3\right). \end{aligned} \tag{28}$$

Without calculating the isotropic invariants, we can see frome the compenents of $\mathbf{D}$ that the condition of incompressibility of our flow is given by:

$$\frac{\partial^2}{\partial x^2}\dot{g}_1 + \frac{\partial^2}{\partial y^2}\dot{g}_2 + \frac{\partial^2}{\partial z^2}\dot{g}_3 = 0. \tag{29}$$

The previous equation also means that the volume does not vary, that is to say $\rho(x,t) = \rho_0$.
Applying these conditions to the Navier-Stokes equations, they become:

$$\begin{aligned} \rho_0 v_0 \ddot{g}_1 + \frac{\partial p}{\partial x} &= 2\mu v_0 \frac{\partial \dot{g}_1}{\partial x} + F_x; \\ \rho_0 v_0 \ddot{g}_2 + \frac{\partial p}{\partial y} &= 2\mu v_0 \frac{\partial \dot{g}_2}{\partial y} + F_y; \\ \rho_0 v_0 \ddot{g}_3 + \frac{\partial p}{\partial z} &= 2\mu v_0 \frac{\partial \dot{g}_3}{\partial z} + F_z. \end{aligned} \tag{30}$$

Using the following expressions with (48) we obtain components of the partial derivatives of the pressure in the general case by which are:

$$\begin{aligned} \frac{\partial p}{\partial x} &= 2\mu v_0 \frac{\partial \dot{g}_1}{\partial x} - \rho_0 v_0 \ddot{g}_1 + F_x; \\ \frac{\partial p}{\partial y} &= 2\mu v_0 \frac{\partial \dot{g}_2}{\partial y} - \rho_0 v_0 \ddot{g}_2 + F_y; \\ \frac{\partial p}{\partial z} &= 2\mu v_0 \frac{\partial \dot{g}_3}{\partial z} - \rho_0 v_0 \ddot{g}_3 + F_z. \end{aligned} \tag{31}$$

This gives us the general expression of the pressure which becomes:

$$\begin{aligned} p = \frac{1}{3}\left(2\mu v_0 \left(\dot{g}_1 + \dot{g}_2 + \dot{g}_3\right)\right) + \frac{1}{3}\int \left(-v_0\rho_0\ddot{g}_1 + F_x\right)\partial x \\ + \frac{1}{3}\left(\int \left(-v_0\rho_0\ddot{g}_2 + F_y\right)\partial y + \left(-v_0\rho_0\ddot{g}_3 + F_z\right)\partial z\right). \end{aligned} \tag{32}$$

In the case of our kinematics with the component of the speed dependent on time, the rotational is is given by:

$$\begin{aligned} \overrightarrow{\Omega} = v_0\left(\frac{\partial \dot{g}_3}{\partial y} - \frac{\partial \dot{g}_2}{\partial z}\right)\overrightarrow{e_x} + v_0\left(\frac{\partial \dot{g}_1}{\partial z} - \frac{\partial \dot{g}_3}{\partial x}\right)\overrightarrow{e_y} \\ + v_0\left(\frac{\partial \dot{g}_2}{\partial x} - \frac{\partial \dot{g}_1}{\partial y}\right)\overrightarrow{e_z}. \end{aligned} \tag{33}$$

Here a condition which will give us an irrotational flow is that of the initial speed v_0, which means that there is no transformation, ie no flow.
The other conditions for an irrotational flow are given by:

$$\frac{\partial g_1}{\partial t} = \frac{\partial g_2}{\partial t} = \frac{\partial g_3}{\partial t} = 0; \tag{34}$$

or

$$\frac{\partial g_1}{\partial z} = \frac{\partial g_3}{\partial x};$$
$$\frac{\partial g_2}{\partial x} = \frac{\partial g_1}{\partial y}. \tag{35}$$

The divergence becomes:

$$div\left(\overrightarrow{v}\right) = v_0\left(\frac{\partial \dot{g}_1}{\partial x} + \frac{\partial \dot{g}_2}{\partial y} + \frac{\partial \dot{g}_3}{\partial z}\right). \tag{36}$$

This result shows that our flow can be performed with a change in volume. So if there is a transformation with $v_0 \neq 0$, the only condition of incompressibility will be given by:

$$\frac{\partial \dot{g}_1}{\partial x} + \frac{\partial \dot{g}_2}{\partial y} + \frac{\partial \dot{g}_3}{\partial z} = 0. \tag{37}$$

3.1 Case of a speed independent of time

Let's consider a fluid particle located inside a velocity field vortex given by the following kinematics [9]:

$$v_x = -v_0\frac{y}{a}; \;\; v_y = v_0\frac{x}{a}; \;\; v_z = Z; \tag{38}$$

with v_0 and a the initial velocity and the initial radius respectively.
With the previous kinematics, the speed tensor becomes:

$$D_{ij} = 1 \;\; si \;\; i = j = 3$$
$$D_{ij} = 0 \;\; sinon. \tag{39}$$

And its adjoint $\mathbf{D}^*$ is the matrice defined by:

$$D^*_{ij} = 0 \;\; \forall 1 \leq i, j \leq 3 \tag{40}$$

For our study, we will restrict ourselves to isotropic invariants

$$I_1 = 1;$$
$$I_2 = 0; \tag{41}$$
$$I_3 = 0;$$

By seeing the value of the first or the third isotropic invariant, we see the hypothesis of a compressible flow. A variation in volume which will also cause a variation in pressure.
And as the volume varies, we will have the density which varies:

$$\rho = \rho\left(x, t\right); \tag{42}$$

In the simple case where $du_t\left(x\right) = \rho\left(x, t\right)$ we can easily determine the components according to each direction of ρ by:

$$\rho_x = v_x;$$
$$\rho_y = v_y; \tag{43}$$
$$\rho_z = v_z.$$

As a first result, this means that the further we move away from the center of the vortex, the more the density will increase and an increase in density results in a decrease in the volume of the fluid particle.
As a second result we can interpret using Bernoulli's theorem that the pressure decreases with increasing speed, i.e. the further the particle moves from the center the less pressure it is subjected to.
Using the assumption that (17) on ρ et v we obtain:

$$\begin{aligned} \rho_x &= v_x; \\ \rho_y &= v_y; \\ \rho_z &= v_z. \end{aligned} \tag{44}$$

With the Navier-Stokes equations defined in (24) and in the absence of external forces, we obtain the following PDE:

$$\begin{aligned} \frac{\partial p}{\partial x} &= 0; \\ \frac{\partial p}{\partial y} &= 0; \\ \frac{\partial p}{\partial z} &= 0. \end{aligned} \tag{45}$$

This means that the pressure is not a function of any of the three Eulerian variables x, y and z. And as the pressure decreases with the increase in distance (radius a) between the center of the vortex and the position of the fluid particle we find the expression of the pressure which becomes

$$p = \frac{k}{a}; \tag{46}$$

with k a constant.
This means that in the absence of external forces, the pressure p follows an affine variation which increases when the radius a decreases.

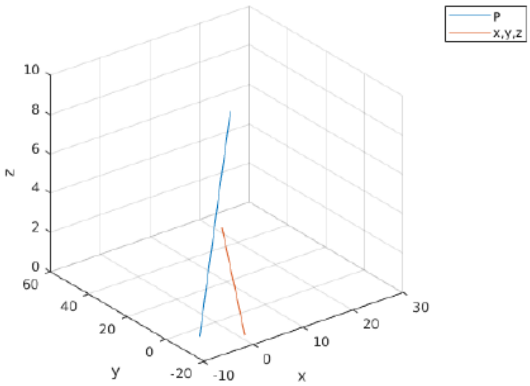
With the relation (24) and in the presence of an external force of component F_x, F_y and F_z, we end up with:

$$\begin{aligned} \frac{\partial p}{\partial x} &= F_x; \\ \frac{\partial p}{\partial y} &= F_y; \\ \frac{\partial p}{\partial z} &= F_z. \end{aligned} \tag{47}$$

In the case where the components of the external force are constants or independent of x, y and z, the equations (41) give us:

$$p = xF_x + yF_y + zF_z + C \tag{48}$$

with C a constant.

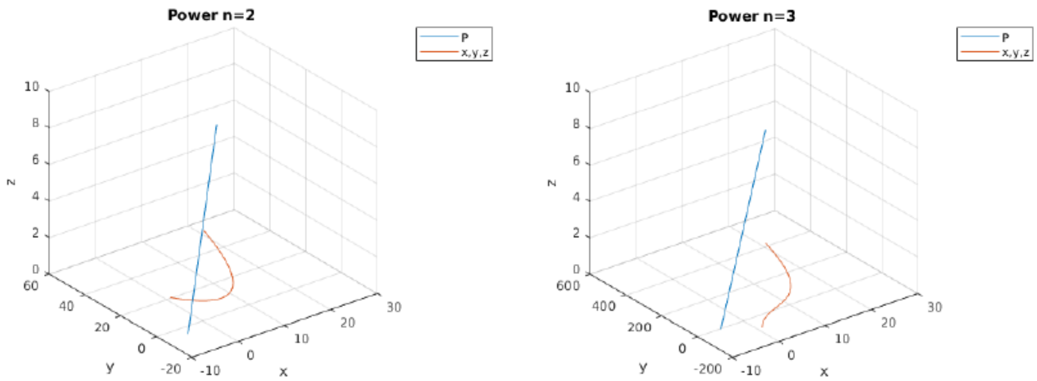


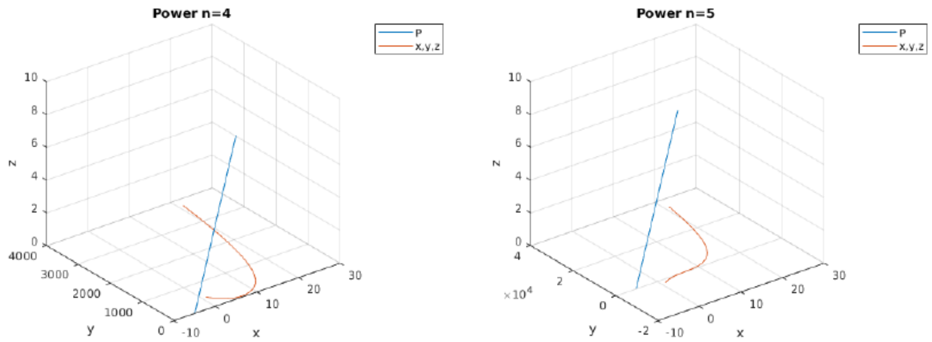
as the expression of the pressure indicates it we have a linear pace.
But on the other hand if the components of the external force are functions of

the three variables x, y and z, then we obtain the expression of the pressure given by:

$$p = \frac{1}{3}\int (F_x\partial x + F_y\partial y + F_z\partial z) \tag{49}$$

We will denote by n, the sum of the powers of a monomial being the greatest in the expression of the pressure.





Even with the components of the external forces which are functions of the variables x, y and z; we have the pressure which follows a increasing linear behavior for a power from $n = 2$ to $n = 5$. We notice a resemblance of pressure and of the trajectory according to whether n is even or odd.

We find that in the case where $du_t(x) = \rho(x,t)$, the fact that the density is a constant or is dependent on the position of the fluid particle does not influence the pressure component. this is due to the fact that the speed components are not time dependent.

Using the relation (39) the value of the rotational becomes

$$\overrightarrow{\Omega} = 2\frac{v_0}{a}\overrightarrow{e_z}. \tag{50}$$

Using also the relation (42), the divergence is given by:

$$div\left(\overrightarrow{V}\right) = v_0 \tag{51}$$

this value of the divergence comes to prove again the variation of the volume of the particle.

3.2 Case of a speed dependent of time

Let us now consider a fluid particle inside a vortex with a helical trajectory discribed by the following kinematic [9]:

$$\begin{aligned} x = Rcos\left(\Theta t\right)\varepsilon\left(Z\right) \\ ;\ \ y = Rsin\left(\Theta t\right)\varepsilon\left(Z\right);\ \ z = Zt; \end{aligned} \tag{52}$$

where R is the initial radius, Θ the angle before deformation and $\varepsilon = \varepsilon(Z)$ a function of Z representing here the perturbation parameter.
From this kinematics, we can find the particule speed components by derivation of the deformation kinematic components according to the time. what gives:

$$\begin{aligned} v_x = -\Theta Rsin\left(\Theta t\right)\varepsilon\left(Z\right) \\ ;\ \ v_y = \Theta Rcos\left(\Theta t\right)\varepsilon\left(Z\right);\ \ v_z = Z; \end{aligned} \tag{53}$$

The speed tensor becomes:

$$\begin{aligned} D_{11} = cos\left(\Theta t\right)\varepsilon; D_{22} = \Theta Rcos\left(\Theta t\right)\varepsilon; \\ D_{33} = 1; \\ D_{12} = D_{21} = \frac{1}{2}\Theta\left(1 - R\Theta\right)Rsin\left(\Theta t\right)\varepsilon; \\ D_{13} = D_{31} = -\frac{1}{2}\varepsilon' Rcos\left(\Theta t\right); \\ D_{23} = D_{32} = \frac{1}{2}\varepsilon' Rcos\left(\Theta t\right). \end{aligned} \tag{54}$$

In order to find a condition of incompressibility, we compute the first isotropic invariant:

$$I_1 = \varepsilon\left(1 + \Theta R\right) cos\left(\Theta t\right) + 1. \qquad (55)$$

So a condition of incompressibility is given by:

$$t = \frac{1}{\Theta} cos^{-1}\left(-\frac{1}{\varepsilon\left(1 + \Theta R\right)}\right). \qquad (56)$$

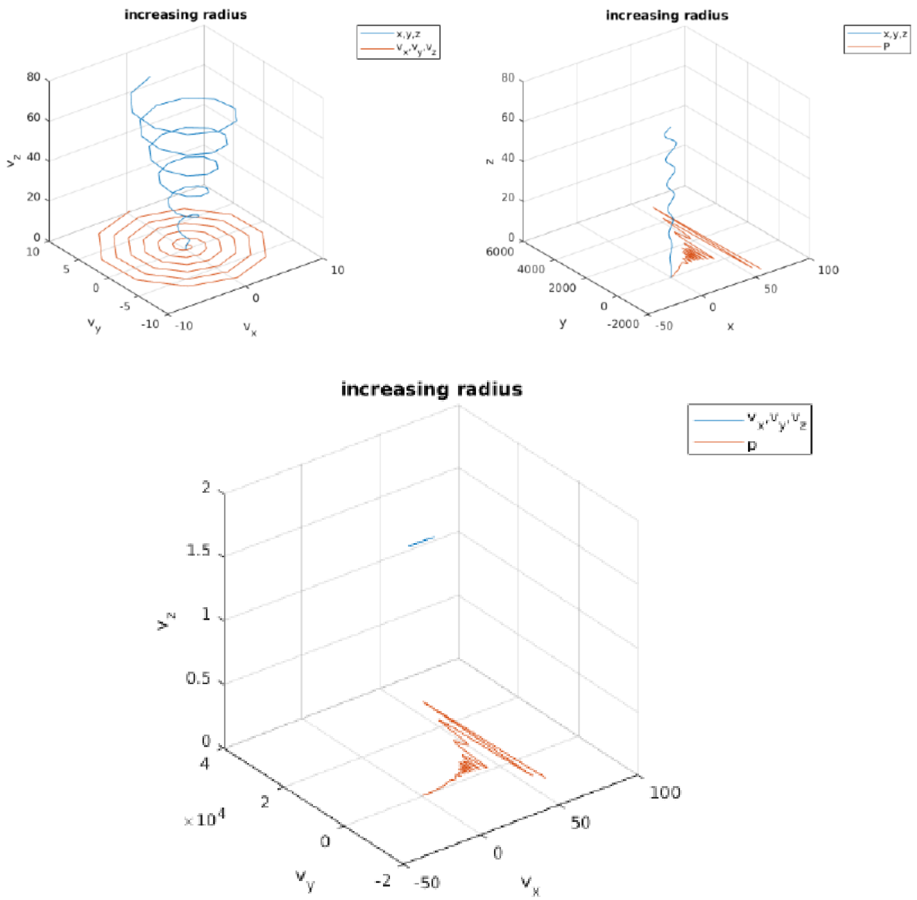
As in the previous case, we have a pressure defined by:

$$p = \frac{1}{3}\int\left(F_x \partial x + F_y \partial y + F_z \partial z\right) \qquad (57)$$

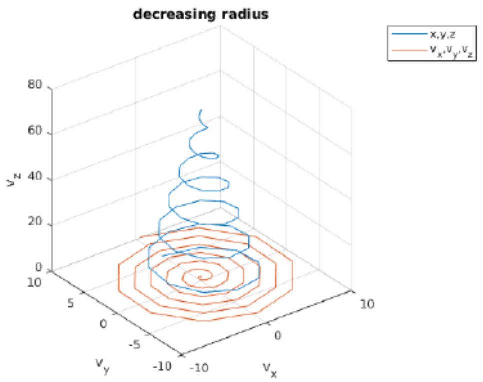
The use of the previous relation allows us to have the pressure from the kinematics. So then:

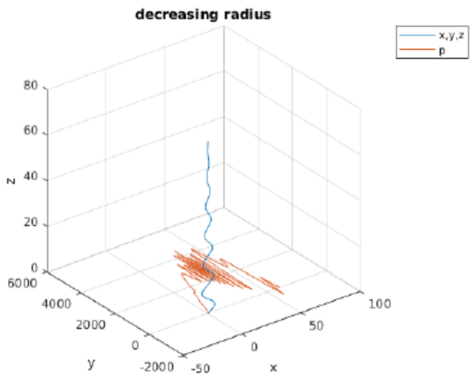
$$\begin{aligned} p = C + \frac{2}{3}\mu v_0\left(R\Theta\left(cos\left(\Theta t\right) - sin\left(\Theta t\right)\right)\right) \\ +\frac{1}{3}\left(2\mu v_0 Z + R\left(F_x - \frac{1}{2}\rho_0 R\Theta^2 cos\left(\Theta t\right)\right)\right) \\ +\frac{1}{3}\left(ZF_z - R\left(\Theta cos\left(\Theta t\right) + \frac{1}{\Theta} sin\left(\Theta t\right)\right)\right). \end{aligned} \qquad (58)$$

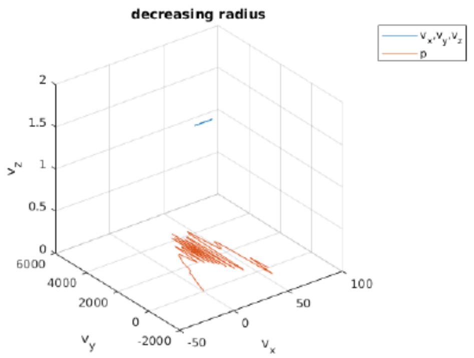
with C representing a costant of integration, v_0 is the initial speed and μ defines the viscosity
In the case of a vortex with a radius which evolves in an increasing way with time, the simulations of the flow kinematics, the flow speed and the flow pressure give the following graphs:



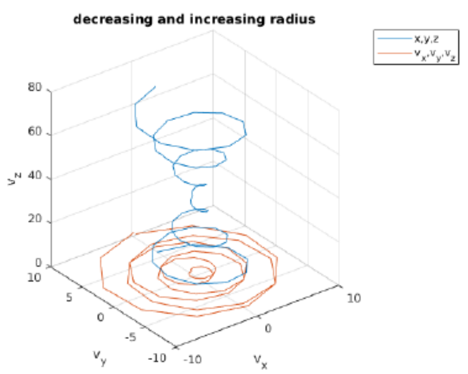
In the case of a vortex with a radius which evolves in a decreasing way with time, the simulations of the flow kinematics, the flow speed and the flow pressure give the following graphs:

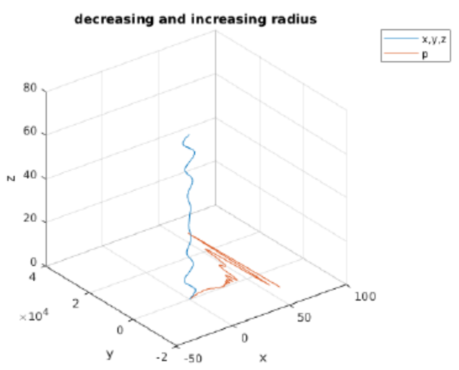


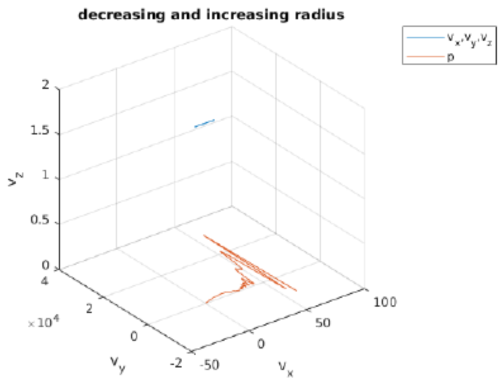




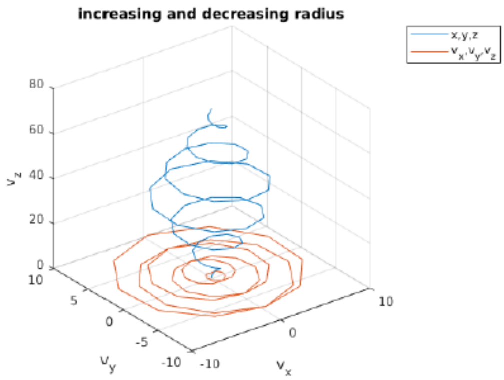
In the case of a vortex with a radius which evolves in a decreasing way and then in an increasing way with time, the simulations of the flow kinematics, the flow speed and the flow pressure give the following graphs:

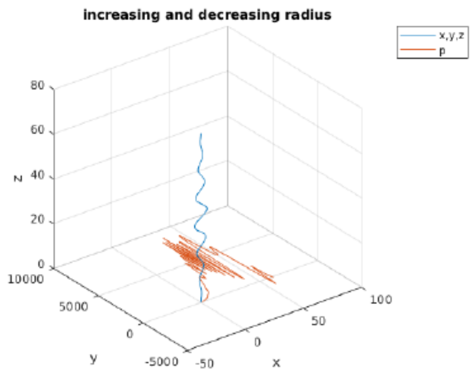


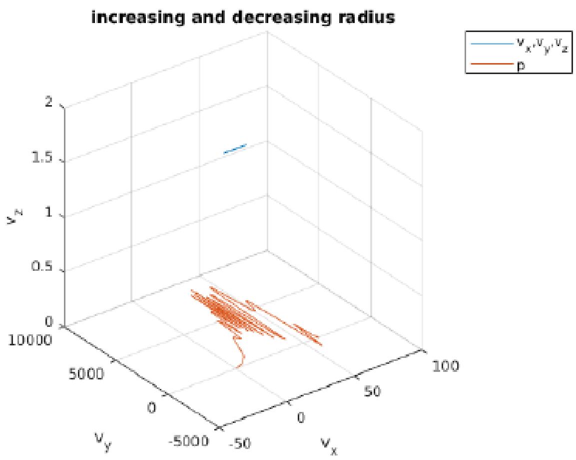




In the case of a vortex with a radius which evolves in an increasing way and then in a decreasing way with time, the simulations of the flow kinematics, the flow speed and the flow pressure give the following graphs:







As an important result in our simulation, we can see that the behavior of the velocity remains in a different evolution which does not show the sinusoidal shape in a different domain with a rectilinear pace on the scale of variation of the trajectory and the pressure. We can also note that the difference in trajectory leads to a difference in behavior in terms of pressure. This proves that the trajectory followed by the particle influences the pressure undergone by the particle, which in practice shows a good behavior of our models. Nevertheless, we can see a small resemblance of the pressure between the model of increasing radius and the model of decreasing then increasing radius but also between the model of decrising radius and the model of increasing then decreasing radius. Using the relation (39) the value of the rotational becomes

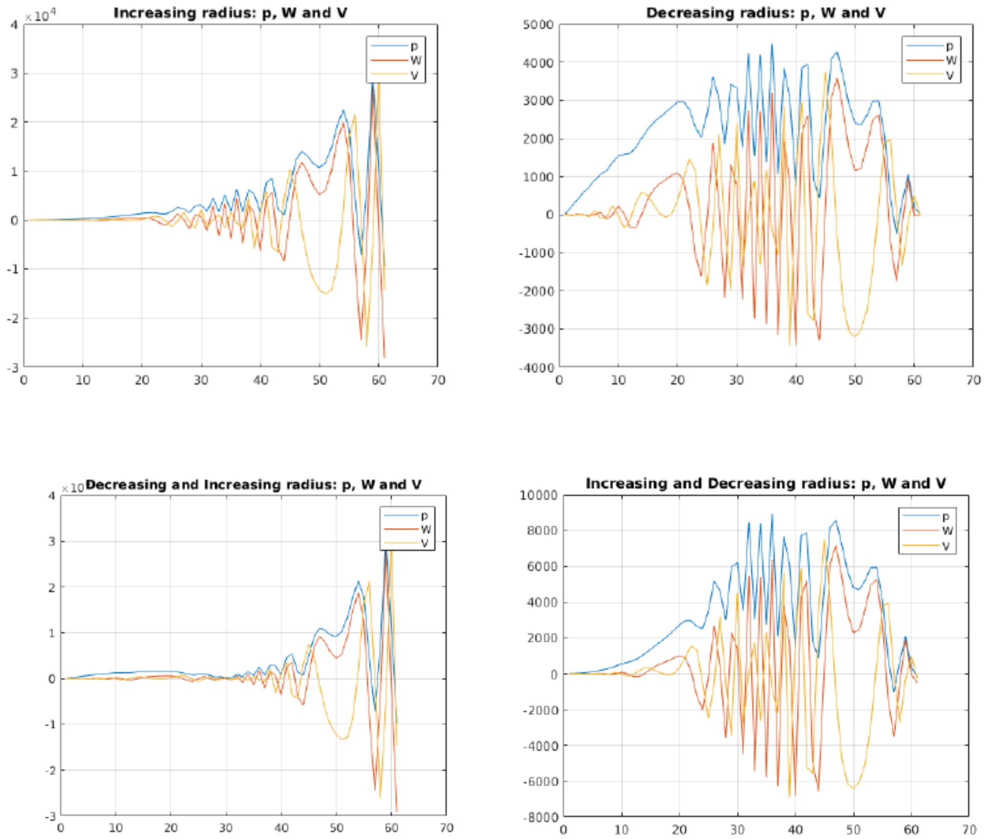
$$\mathbf{W} = \overrightarrow{\Omega} = \left(\Theta\left(1 - tR\right)cos\left(\Theta t\right) - Rsin\left(\Theta t\right)\right)\overrightarrow{e_z}. \tag{59}$$

Using also the relation (42), the divergence is given by:

$$\mathbf{V} = div\left(\overrightarrow{v}\right) = Rcos\left(\Theta t\right) - \Theta\left(1 + tR\right)sin\left(\Theta t\right) \tag{60}$$

this value of the divergence shows that in the case of our kinematics where the velocity components depend on the time, it is possible to have a non variation of the volume of the particle in certain conditions.

To better visualize the rotational and the divergence, We simulate them together with the pressure



The four figures show how the three simulated variables vary differently in each case of trajectory. We can see that the variables are in no case superimposable between two models. This means that the trajectory of the fluid particle has an influence on these variables. Nevertheless, we can see also a small resemblance between the model increasing radius and the model decreasing then increasing radius but also between the model decrising radius and the model increasing then decreasing radius.

4 Results

In this mathematical study of the modeling of the behavior of a particle inside a vortex, the mathematical calculations and simulations of the trajectory, the speed, the pressure, the ratational and the divergence gave us: In the case of a transformation where the speed components are independent on the time, we have:

- The hypothesis of an incompressible flow is impossible. A variation of the volume that goes to cause variation of the density and the pressure.
- The further away the particle is from the center, the more volume and speed will increase and the pressure will decrease. This result shows that near the center of the vortex a particle is compressed, which is dangerous for a living being.
- In the absence of external forces, the pressure is not a function of any of the three Eulerian variables x, y and z and depends only on the radius.
- In the case where $du_t(x,t) = \rho(x,t)$, the fact that the density is a constant or is dependent on the position of the particle fluid does not affect the pressure. And that we get the same pressure even though $du_t(x,t) \neq \rho(x,t)$, because quite simply, if the speed is independent of time, the value of the density does not change the pressure in any way.
- The simulation of the pressure and the trajectory with the components of the external forces which are functions of the variables x, y and z shows that we have the pressure which follows a increasing linear behavior for a power from $n = 2$ to $n = 5$. We notice a resemblance of pressure and of the trajectory according to whether that n is even or odd.

In the case of a transformation where the speed components dependent on the time variable, we have:

- The hypothesis of an incompressible flow is possible in certain conditions verified by the partial derivatives or by a certain value of of the time t. In the case of a constant volume density and in the presence of an external force, we have the Navier-Stock equations which depend on many parameters. The calculation of the pressure gives $p = p(R, \Theta, Z, t)$.
- The result proves that we have here a pressure which depends on the position of the particle. Consequently the particle will undergo variations in volume.
- The expression of the divergence of the fluid particle comes to prove again the possibility of having an incompressible flow.
- In our simulation, we see that the behavior of the velocity remains in a different evolution which does not show the sinusoidal shape in a different domain with a rectilinear pace on the scale of variation of the trajectory and the pressure.
- We even see that these four speeds appear to be stackable between the different models, what is false and also note that the difference in trajectory leads to a difference in behavior in terms of pressure. This proves that the trajectory followed by the particle influences the pressure undergone by the particle, which in practice shows a good behavior of our model.

Nevertheless, we see a small resemblance of the pressure between the model of increasing radius and the model of decreasing then increasing radius but also between the model of decrising radius and the model of increasing then decreasing radius.

- The simulation of ratatinal, divergence and pressure together shows that variables vary differently in each case of trajectory. We can see that the variables are in no case superimposable between two models. This means that the trajectory of the fluid particle has an influence on these variables.
- Nevertheless, we see also a small resemblance between the model increasing radius and the model decreasing then increasing radius but also between the model decrising radius and the model increasing then decreasing radius.

This study shows that if the flow kinematics have velocity components which depend on time, then the pressure, the rotational and the divergence will all depend on the trajectory, and shows also that the further away we are from a vortex the less we feel its effect.

5. Conclusion

In this document, we have proposed to study the the behavior of a fluid particle inside a vortex in cases where the speed component depends or not on the time, in order to better understand its behavior according to the boundary conditions. After preliminary, we have difined flow kinematics and / or velocity fields, in order to calculate the velocity tensor, isotropic invariants, pressure, rotational, divergence and incompressibility conditions depending on whether the velocity is dependent on the time or not. We have shown that in the case where the speed is independent of time, it is impossible to have a situation of incompressibility except in the case where there is no flow and the value of the density does not influence on the pressure of the particle and that this pressure is independant on the position where the particle is located.

We have also shown that in the case where the speed is dependent on time, it is possible to have a situation of incompressibility if the partial differential equations satisfy certain conditions or for a certain value of the parameter t. In this case the density has a strong influence on the pressure of the particle which strongly depend on the position where the particle is located.
The simulation in the case that $v \neq v(t)$ of the pressure and the trajectory with the components of the external forces which are functions of the variables x, y and z shows that we have the pressure which follows a increasing linear behavior for a power from $n = 2$ to $n = 5$. We notice a resemblance of pressure and of the trajectory according to whether that n is even or odd.
The simulation in the case that $v = v(t)$ shows that the behavior of the velocity remains in a different evolution which does not show the sinusoidal shape but

a rectilinear pace. It even shows that these four speeds appear to be stackable between the different models but it is not true in reality and we also note that the difference in trajectory leads to a difference in behavior in terms of pressure. So the trajectory followed by the particle influences the pressure undergone by the particle, what shows a good behavior of our model. We have a small resemblance of the pressure between the models two by two. The simulation of ratatinal, divergence and pressure together shows difference in each case of trajectory and the variables are in no case superimposable between two models. So the trajectory of the fluid particle has an influence on these variables. A small resemblance have been seen between the models two by two.

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