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**Research Paper**

# Some Limit Problems Involving Fractional Functions

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***ABSTRACT:*** *Local fractional calculus plays an important role in this paper. We mainly use the local fractional integrals of three fractional functions to solve three types of limit problems.*
***KEYWORDS:*** *Local Fractional Calculus, Local Fractional Integrals, Fractional Functions, Limit Problems.*

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## I. INTRODUCTION

The concept of local fractional calculus has always been of interest to mathematicians, physicists and engineers [1-7]. Local fractional calculus is a natural generalization of classical differentiation and integration from traditional functions to fractional functions [8-9]. In this article, local fractional calculus plays an important role. Using the local fractional integrals of some fractional functions, we can easily solve the following three types of limit problems:

$$\lim_{n\to\infty}\frac{E_\alpha\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+E_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+E_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha}, \tag{1}$$

$$\lim_{n\to\infty}\frac{cos\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+cos_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+cos_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha}, \tag{2}$$

and

$$\lim_{n\to\infty}\frac{sin_\alpha\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+sin_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+sin_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha}, \tag{3}$$

where $0<\alpha\le 1$, $r$ is a real number, and $m$ is a positive integer. $E_\alpha$, $cos_\alpha$, $sin_\alpha$ are $\alpha$-fractional exponential function, cosine function, and sine function respectively.

## II. DEFINITIONS AND PROPERTIES

Firstly, we introduce fractional integrals of three fractional functions and their properties.

**Definition 2.1** ([8])**:** Suppose that $0<\alpha\le 1$, and $x$ is a real variable. The $\alpha$-fractional exponential function, cosine function and sine function are defined as follows:

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{1}$$

$$cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{2}$$

and

$$sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \tag{3}$$

**Definition 2.2** ([9])**:** Suppose that $0<\alpha\le 1$, and $u:[a,b]\to R$. If

$$\lim_{\|\Delta\|\to 0}\sum_{k=1}^{n}u(\xi_k)\,(x_k-x_{k-1})^\alpha \tag{4}$$

exists, then we say that $u$ is a $\alpha$-fractional Riemann integrable function on $[a,b]$. The partitions of the interval $[a,b]$ are $[x_{k-1},x_k]$, $k=1,\cdots,n$, and $x_0=a$, $x_n=b$, $\xi_k\in[x_{k-1},x_k]$, $\Delta x_k=x_k-x_{k-1}$, $\|\Delta\|=\max_{k=1,\cdots,n}\{\Delta x_k\}$. The limit is denoted by

$$\lim_{\|\Delta\|\to 0}\sum_{k=1}^{n} u(\xi_k)\,(x_k - x_{k-1})^{\alpha} = \int_a^b u(x)\,dx^{\alpha}, \tag{5}$$

which is called the $\alpha$-fractional Riemann integral (or local $\alpha$-fractional integral) of $u$ on $[a,b]$.

**Definition 2.3** ([9])**:** Assume that $0<\alpha\le 1$, $(-1)^{\alpha}=-1$, $u:[a,b]\to R$ and $x_0\in(a,b)$. $u$ is called local $\alpha$-fractional differentiable at $x_0$ if $\lim\limits_{x\to x_0}\frac{u(x)-u(x_0)}{(x-x_0)^{\alpha}}$ exists. The $\alpha$-fractional derivative of $u(x)$ at $x_0$ is denoted by

$$u^{(\alpha)}(x_0) = \frac{du}{dx^{\alpha}}(x_0) = \Gamma(\alpha+1)\cdot\lim_{x\to x_0}\frac{u(x)-u(x_0)}{(x-x_0)^{\alpha}}, \tag{6}$$

where $\Gamma(\ \ )$ is the gamma function. If $u$ is local $\alpha$-fractional differentiable at any point in open interval $(a,b)$, then we say that $u$ is a local $\alpha$-fractional differentiable function on $(a,b)$. Moreover, the $n$-th order local $\alpha$-fractional derivative $\left(u^{(\alpha)}\right)\left(u^{(\alpha)}\right)\cdots\left(u^{(\alpha)}\right)(x)$ of $u(x)$, is denoted by $u^{(n\alpha)}(x)$ or $\frac{d^n u}{(dx^{\alpha})^n}(x)$, where $n$ is a positive integer, and let $u^{(0)}(x)=u(x)$.

The followings are the local $\alpha$-fractional derivatives of three $\alpha$-fractional functions.

**Proposition 2.4:** *Suppose that* $0<\alpha\le 1$, *then*

$$[E_{\alpha}(x^{\alpha})]^{(\alpha)}(x) = E_{\alpha}(x^{\alpha}), \tag{7}$$

$$[sin_{\alpha}(x^{\alpha})]^{(\alpha)}(x) = cos_{\alpha}(x^{\alpha}), \tag{8}$$

$$[cos_{\alpha}(x^{\alpha})]^{(\alpha)}(x) = -sin_{\alpha}(x^{\alpha}). \tag{9}$$

**Theorem 2.5** ([9])**:** (fundamental theorem of fractional analytic functions for local fractional calculus)**:** *Let* $0<\alpha\le 1$, $(-1)^{\alpha}=-1$. *If* $f_{\alpha}:[a,b]\to R$ *and* $F_{\alpha}:[a,b]\to R$ *are* $\alpha$*-fractional analytic functions such that* ${F_{\alpha}}^{(\alpha)}(x^{\alpha}) = f_{\alpha}(x^{\alpha})$ *for all* $x\in(a,b)$, *then*

$$\int_a^b f_{\alpha}(x^{\alpha})\,dx^{\alpha} = \Gamma(\alpha+1)\left(F_{\alpha}(b^{\alpha}) - F_{\alpha}(a^{\alpha})\right). \tag{10}$$

In the following, we obtain the indefinite $\alpha$-fractional integrals involving three $\alpha$-fractional functions.

**Proposition 2.6** ([9])**:** *If* $0<\alpha\le 1$, *then*

$$\int E_{\alpha}(x^{\alpha})\,dx^{\alpha} = \Gamma(\alpha+1)E_{\alpha}(x^{\alpha}) + C, \tag{11}$$

$$\int cos_{\alpha}(x^{\alpha})\,dx^{\alpha} = \Gamma(\alpha+1)sin_{\alpha}(x^{\alpha}) + C, \tag{12}$$

$$\int sin_{\alpha}(x^{\alpha})\,dx^{\alpha} = -\Gamma(\alpha+1)cos_{\alpha}(x^{\alpha}) + C. \tag{13}$$

## III. RESULTS AND EXAMPLES

In this section, we will solve the limit problems involving three fractional functions. On the other hand, several examples are provided to illustrate our results.

**Theorem 3.1:** *Let* $0<\alpha\le 1$, $r$ *be a real number, and* $m$ *be a positive integer. Then*

$$\lim_{n\to\infty}\frac{E_{\alpha}\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+E_{\alpha}\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+E_{\alpha}\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^{\alpha}} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{r^k\cdot\Gamma(km\alpha+1)}{\Gamma(k\alpha+1)\cdot\Gamma((km+1)\alpha+1)}, \tag{14}$$

$$\lim_{n\to\infty}\frac{cos\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+cos_{\alpha}\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+cos_{\alpha}\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^{\alpha}} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}\cdot\Gamma(2km\alpha+1)}{\Gamma(2k\alpha+1)\cdot\Gamma((2km+1)\alpha+1)}, \tag{15}$$

*and*

$$\lim_{n\to\infty}\frac{sin_{\alpha}\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+sin_{\alpha}\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+sin_{\alpha}\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^{\alpha}} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}\cdot\Gamma((2km+m)\alpha+1)}{\Gamma((2k+1)\alpha+1)\cdot\Gamma((2km+m+1)\alpha+1)}. \tag{16}$$

**Proof** Since $E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}$, it follows that $E_{\alpha}(rx^{m\alpha}) = \sum_{k=0}^{\infty}\frac{r^k x^{km\alpha}}{\Gamma(k\alpha+1)}$, and hence by fundamental theorem of fractional analytic functions for local fractional calculus, the indefinite $\alpha$-fractional integral

$$\int E_{\alpha}(rx^{m\alpha})\,dx^{\alpha} = \int\sum_{k=0}^{\infty}\frac{r^k x^{km\alpha}}{\Gamma(k\alpha+1)}dx^{\alpha} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{r^k\cdot\Gamma(km\alpha+1)\cdot x^{(km+1)\alpha}}{\Gamma(k\alpha+1)\cdot\Gamma((km+1)\alpha+1)} + C. \tag{17}$$

Therefore, the definite $\alpha$-fractional integral

$$\int_0^1 E_\alpha(rx^{m\alpha})\,dx^\alpha = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{r^k\cdot\Gamma(km\alpha+1)}{\Gamma(k\alpha+1)\cdot\Gamma((km+1)\alpha+1)}. \tag{18}$$

Since

$$\int_0^1 E_\alpha(rx^{m\alpha})\,dx^\alpha = \lim_{n\to\infty}\frac{E_\alpha\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+E_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+E_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha}. \tag{19}$$

It follows that

$$\lim_{n\to\infty}\frac{E_\alpha\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+E_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+E_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{r^k\cdot\Gamma(km\alpha+1)}{\Gamma(k\alpha+1)\cdot\Gamma((km+1)\alpha+1)}.$$

On the other hand, by $cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}$, we have $cos_\alpha(rx^{m\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k} x^{2km\alpha}}{\Gamma(2k\alpha+1)}$. It follows that

$$\int cos_\alpha(rx^{m\alpha})\,dx^\alpha = \int\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}x^{2km\alpha}}{\Gamma(2k\alpha+1)}\,dx^\alpha = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}\cdot\Gamma(2km\alpha+1)\cdot x^{(2km+1)\alpha}}{\Gamma(2k\alpha+1)\cdot\Gamma((2km+1)\alpha+1)}+C. \tag{20}$$

Thus,

$$\int_0^1 cos_\alpha(rx^{m\alpha})\,dx^\alpha = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}\cdot\Gamma(2km\alpha+1)}{\Gamma(2k\alpha+1)\cdot\Gamma((2km+1)\alpha+1)}. \tag{21}$$

And hence,

$$\lim_{n\to\infty}\frac{cos\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+cos_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+cos_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}\cdot\Gamma(2km\alpha+1)}{\Gamma(2k\alpha+1)\cdot\Gamma((2km+1)\alpha+1)}.$$

Also, using $sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}$ yields $sin_\alpha(rx^{m\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}x^{(2km+m)\alpha}}{\Gamma((2k+1)\alpha+1)}$. Thus,

$$\begin{aligned}\int sin_\alpha(rx^{m\alpha})\,dx^\alpha &= \int\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}x^{(2km+m)\alpha}}{\Gamma((2k+1)\alpha+1)}\,dx^\alpha\\ &= \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}\cdot\Gamma((2km+m)\alpha+1)\cdot x^{(2km+m+1)\alpha}}{\Gamma((2k+1)\alpha+1)\cdot\Gamma\left((2km+m+1)\alpha+1\right)}+C.\end{aligned}$$

And hence,

$$\int_0^1 sin_\alpha(rx^{m\alpha})\,dx^\alpha = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}\cdot\Gamma((2km+m)\alpha+1)}{\Gamma((2k+1)\alpha+1)\cdot\Gamma\left((2km+m+1)\alpha+1\right)}.$$

Therefore,

$$\lim_{n\to\infty}\frac{sin_\alpha\left(r\left(\frac{1}{n}\right)^{m\alpha}\right)+sin_\alpha\left(r\left(\frac{2}{n}\right)^{m\alpha}\right)+\cdots+sin_\alpha\left(r\left(\frac{n}{n}\right)^{m\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}\cdot\Gamma((2km+m)\alpha+1)}{\Gamma((2k+1)\alpha+1)\cdot\Gamma\left((2km+m+1)\alpha+1\right)}.$$

Q.e.d.

In Theorem 3.1, let $r=1$ and $m=1$, we obtain the following results.

**Corollary 3.2:** *If* $0<\alpha\le 1$, *then*

$$\lim_{n\to\infty}\frac{E_\alpha\left(\left(\frac{1}{n}\right)^{\alpha}\right)+E_\alpha\left(\left(\frac{2}{n}\right)^{\alpha}\right)+\cdots+E_\alpha\left(\left(\frac{n}{n}\right)^{\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot[E_\alpha(1)-1], \tag{22}$$

$$\lim_{n\to\infty}\frac{cos\left(\left(\frac{1}{n}\right)^{\alpha}\right)+cos_\alpha\left(\left(\frac{2}{n}\right)^{\alpha}\right)+\cdots+cos_\alpha\left(\left(\frac{n}{n}\right)^{\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot sin_\alpha(1), \tag{23}$$

*and*

$$\lim_{n\to\infty}\frac{sin_\alpha\left(\left(\frac{1}{n}\right)^{\alpha}\right)+sin_\alpha\left(\left(\frac{2}{n}\right)^{\alpha}\right)+\cdots+sin_\alpha\left(\left(\frac{n}{n}\right)^{\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot[1-cos_\alpha(1)]. \tag{24}$$

.

**Example 3.3:** $\lim_{n\to\infty}\frac{E_\alpha\left(2\left(\frac{1}{n}\right)^{3\alpha}\right)+E_\alpha\left(2\left(\frac{2}{n}\right)^{3\alpha}\right)+\cdots+E_\alpha\left(2\left(\frac{n}{n}\right)^{3\alpha}\right)}{n^\alpha} = \Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{2^k\cdot\Gamma(3k\alpha+1)}{\Gamma(k\alpha+1)\cdot\Gamma((3k+1)\alpha+1)}$.

(25)

**Example 3.4:** $\lim_{n\to\infty}\frac{cos\left(4\left(\frac{1}{n}\right)^{2\alpha}\right)+cos_\alpha\left(4\left(\frac{2}{n}\right)^{2\alpha}\right)+\cdots+cos_\alpha\left(4\left(\frac{n}{n}\right)^{2\alpha}\right)}{n^\alpha}=\Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k 4^{2k}\cdot\Gamma(4k\alpha+1)}{\Gamma(2k\alpha+1)\cdot\Gamma((4k+1)\alpha+1)}.$

(26)

**Example 3.5:** $\lim_{n\to\infty}\frac{sin_\alpha\left(7\left(\frac{1}{n}\right)^{5\alpha}\right)+sin_\alpha\left(7\left(\frac{2}{n}\right)^{5\alpha}\right)+\cdots+sin_\alpha\left(7\left(\frac{n}{n}\right)^{5\alpha}\right)}{n^\alpha}=\Gamma(\alpha+1)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k 7^{2k+1}\cdot\Gamma((10k+5)\alpha+1)}{\Gamma((2k+1)\alpha+1)\cdot\Gamma\left((10k+6)\alpha+1\right)}.$

(27)

## IV. CONCLUSION

Local fractional calculus plays an important role in this study. The main purpose of this article is to deal with three limit problems involving fractional functions. Using the local fractional integrals of three fractional functions, we can easily solve these three limit problems. In the future, we will expand our research field to the applied mathematics and fractional calculus problems.

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