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**Research Paper**

# Stability Of A Heavy Elastica With A One-Parameter Deflection Curve

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***ABSTRACT:*** *This paper deals with a certain method to examine the state of equilibrium of a heavy elastica. Such object is used e.g. for modeling a flat textile structure loaded with its dead weight and axial force. The elastica represents a longitudinal section of a fabric. It was assumed that the elastica rests on a flat, immovable base. Only those forms of deformed elastica were considered where its two ends were supported by pivot bearings, and the tangent at those points lay on the immovable supporting plane. In the analysis, shape of the deflection curve was determined for a given axial force, and it was examined whether a given position is stable or unstable. The analysis was made on the basis of the energetic method, by examining potential energy of the system. The investigations can be used for simulation of fabric buckling, folding and for another applications of textile mechanics.*
***Keywords:*** *stability, elastica, deflection curve, states of equilibrium, bending theory*

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## I. INTRODUCTION

The theory for the elastica was first formulated by Euler [1], who defined the elastica as a slender rod whose curvature at any point is proportional to the local moment experienced. If the deformation is due to self-weight, it is called a heavy elastica. Greenhill [2] first correctly found the stability criteria for a standing uniform heavy cantilever. Various aspects of the uniform heavy elastica cantilever have been reported previously, e.g. [3-9]. The stability for the vertical pointy heavy cantilever, i.e. the tip tapered into a sharp point, was solved by Dinnik [10] in terms of Bessel functions. The stability of a beam under a concentrated force, which is clamped at one end while sliding over a point support at the other, has been studied by Zhang and Yang [11]. The slip-through of a beam under self-weight resting on two point supports has been examined by Chen et al. [12]. In literature, problems of this kind are sometimes referred to as variable-arc-length beams (Chucheepsakul et al. [13]). A boundary setting as described above, however, has not been reported so far. Most of the preceding work deals with beams for which one end is hinged while the other one may slide freely over a frictionless point support. More recently, the studies have been extended to the static and dynamic behavior of beams under self-weight, which have been analyzed both analytically and experimentally (Pulngern et al. [14, [15]). A slightly modified boundary setting with one support elevated above the other has been investigated by Athisakul and Chucheepsakul (2008).

## II. ASSUMPTIONS OF MODEL AND INITIAL EQUATIONS

Let us consider a heavy elastica resting on a fixed, immovable base (Figure 1). Under the action of compressive forces there arise folds on its surface, which remain there due to the occurrence of friction forces.


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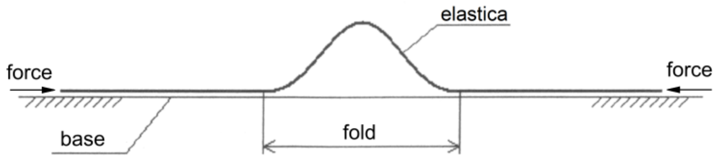


**Figure 1:** An example of elastica deformation in the form of a fold.

Depending on the friction force quantity, those folds will remain or disappear after the action of deforming forces. In order to enable a thorough examination of stability of the deformed elastica, a substitute model was assumed which was limited to its deformed shape, i.e. to the fold.

Let heavy elastica be loaded with the axial force $P$ and continuous load $q$ (linear weight) in the coordinate system as in Figure 2. The elastica rests on a flat, fixed base and is supported on both ends by pivot bearings. It is inextensible so it cannot change its length $l$ under the influence of loads acting on it.
However, it is subject to Hooke's law while being bent, and the known relation for the bending moment $M$ is applicable to it

$$M = EI \frac{1}{\rho}, \quad (1)$$

where $\rho$ stands for the radius of curvature, and $EI$ means the bending rigidity.

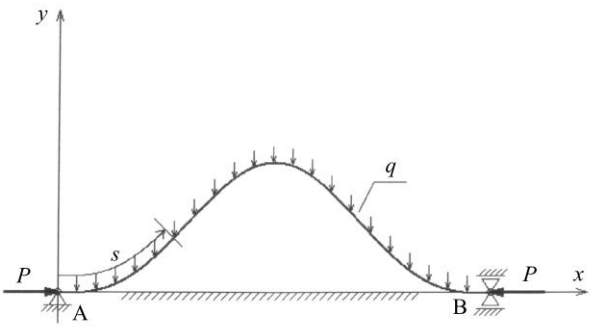


**Figure 2:** The load scheme of elastica in the coordinate system.

No simplifications are applied to the curvature $1/\rho$ as is done with the theory of bending beams, because big deformations are involved here.
In this case, existence of the rigid base causes limitation of the $y$ coordinate. It must be greater or equal to zero for each value of the arc coordinate $s$, which is measured along the deflection curve.
The boundary conditions for this load scheme are the following

$$\begin{aligned} y = 0\big|_{s=0}, \quad y = 0\big|_{s=l}, \\ M = 0\big|_{s=0}, \quad M = 0\big|_{s=l}. \end{aligned} \quad (2)$$

Zero moment $M$ at the points of support A and B results from the fact that apart from the fold, the elastica rests flat on the base and its curvature $1/\rho$ amounts to zero. From that fact it also results that the tangent at the points of support must be horizontal. Thus, we get additional boundary conditions, namely

$$\frac{dy}{ds} = 0\bigg|_{s=0}, \quad \frac{dy}{ds} = 0\bigg|_{s=l}. \quad (3)$$

Let us consider the infinitesimal elastica section presented in Figure 3.

As it has been already mentioned, the elastica is inextensible, thus $dx^2 + dy^2 = ds^2$ .
Therefore we have the following geometrical condition:

$$dy / ds < 1 . \quad (4)$$

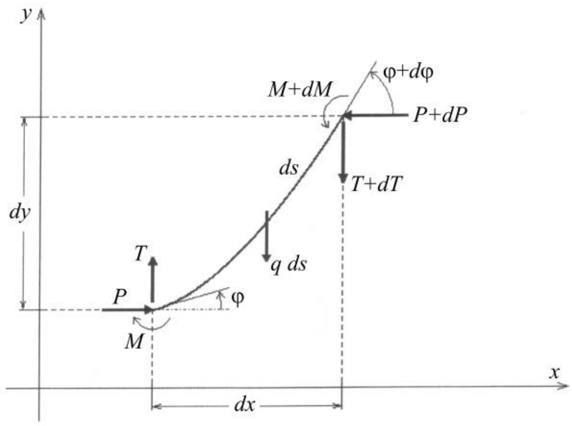


**Figure 3:** The infinitesimal section of elastica.

Now let us write the elementary equations of equilibrium (Figure 3).

$$\begin{aligned} - dP &= 0 , \\ - dT - q\,ds &= 0 , \qquad (5) \\ dM - T\,dx + P\,dy &= 0 . \end{aligned}$$

Basing on the above equations of equilibrium we get the principle of virtual work on the virtual displacements δ*x*, δ*y*, δ*φ*. To do this, we multiply the Equations (5) by appropriate virtual displacements. Then, by adding the sides and integrating within the limits from 0 to *l* we obtain

$$- dP\,\delta x - dT\,\delta y - q\,ds\,\delta y + dM\,\delta\varphi - T\,dx\,\delta\varphi + P\,dy\,\delta\varphi = 0 , \qquad (6)$$

$$- \int_0^l \frac{dP}{ds}\delta x\,ds - \int_0^l \frac{dT}{ds}\delta y\,ds - \int_0^l T\frac{dx}{ds}\delta\varphi\,ds + \int_0^l P\frac{dy}{ds}\delta\varphi\,ds - \int_0^l q\,\delta y\,ds + \int_0^l \frac{dM}{ds}\delta\varphi\,ds = 0 . \quad (7)$$

After integrating by parts we can write

$$\begin{aligned} & - P\,\delta x\Big|_0^l + \int_0^l P\,\delta\left(\frac{dx}{ds}\right)ds - T\,\delta y\Big|_0^l + \int_0^l T\,\delta\left(\frac{dy}{ds}\right)ds - \int_0^l T\frac{dx}{ds}\delta\varphi\,ds + \\ & + \int_0^l P\frac{dy}{ds}\delta\varphi\,ds - \int_0^l q\,\delta y\,ds + M\,\delta\varphi\Big|_0^l - \int_0^l M\,\delta\left(\frac{d\varphi}{ds}\right)ds = 0 . \end{aligned} \qquad (8)$$

Considering that $\frac{dx}{ds} = \cos\varphi$ , $\frac{dy}{ds} = \sin\varphi$ , $\frac{1}{\rho} = \frac{d\varphi}{ds}$ ,

and taking the boundary conditions and the relationship (1), the Equation (8) after reduction takes the form of

$$\int_0^l q\,\delta y\,ds + P\,\delta x_B + \int_0^l \frac{1}{2}EI\,\delta\left(\frac{d\varphi}{ds}\right)^2 ds = 0 . \quad (9)$$

Thus, in the end

$$\delta\left[ q\int_0^l y\,ds + P x_B + \frac{1}{2} EI \int_0^l \left( \frac{d\varphi}{ds} \right)^2 ds \right] = 0 \ . \quad (10)$$

The obtained Equation (9) represents the principle of virtual work, which provides that in the state of equilibrium the sum of work of all actual forces (both external and internal) acting on the system, for any virtual displacements, is equal to zero.
The functional occurring in the square brackets in Equation (10) is total potential energy of the system (potential of external and internal forces).

$$J[y] = q\int_0^l y\,ds + P x_B + \frac{1}{2} EI \int_0^l \left( \frac{d\varphi}{ds} \right)^2 ds \ . \quad (11)$$

The Equation (10) can be thus written in the form of

$$\delta J[y] = 0 \ . \quad (12)$$

This is the necessary condition for existence extremum of the functional $J[y]$.
If equilibrium is stable (stability), then the potential energy reaches minimum in the balance point. In the case of maximum potential energy, we deal however with the unstable state of equilibrium (labile equilibrium) [17], [18].

**2.1 Deflection curve**

As it is already known the deflection curve of the elastica in the state of equilibrium should present the functional extremum (11), respectively minimum for stable equilibrium, and maximum for labile one. In order to determine the functional extremum, let us take for granted the equation of the deflection curve, which fulfils given boundary conditions. Let the deflection curve be described by the Equation (13)

$$y = A \sin\left( \frac{\pi s}{l} \right) + B \sin\left( \frac{3\pi s}{l} \right), \quad (13)$$

where $A$ and $B$ are unknown coefficients for the time being.
It can be easily noticed that the function (13) fulfils the boundary conditions (2).
As regards the additional boundary conditions (3) concerning the derivative $dy/ds$, we obtain from them relationship between the $A$ and $B$ coefficients in the form of $B = -\frac{1}{3} A$ .
Eventually, after transformations, the deflection curve is defined by the equation

$$y = \frac{4}{3} A \sin^3\left( \frac{\pi s}{l} \right). \quad (14)$$

The range of admissible values of $A$ parameter will be presented in the next point.

**2.2. Admissible values for the shape parameter**

We have the deflection curve defined by the Equation (14). The shape parameter $A$ occurring in the equation will be hereafter presented in the dimensionless form $a = A/l$ , related to the length $l$. This parameter cannot take full range of values. Below there is a precise definition of the interval of admissible values of $a$.

In the model assumptions the existence of a fixed base imposes the condition $y \geq 0$ . Thus

$$\frac{4}{3} A \sin^3\left(\frac{\pi s}{l}\right) \geq 0 \;. \qquad (15)$$

Since we discuss only the interval from *s*=0 to *s*=*l*, then from the function curve $\sin^3(\pi s/l)$ within this interval it follows that for the condition (15) to be fulfilled, it must be *A*≥0, that is

$$a \geq 0. \qquad (16)$$

Apart from that, in the model assumptions the condition (4) was given due to inextensibility of the elastica, which concerns the derivative $dy/ds$. Applying it now, and substituting $\xi = \pi s/l$ we obtain that for $0 \leq \xi \leq \pi$

$$4 a \pi \sin^2(\xi)\cos(\xi) < 1 \;. \qquad (17)$$

To find the value *a* from that, first we determine the maximum value of the function within the interval $0 \leq \xi \leq \pi$

$$f = \sin^2(\xi)\cos(\xi)\;, \qquad (18)$$

because to satisfy the inequality (17) it is sufficient to substitute the function (18) with its maximal value.
On examining the function (18) it can be proved that in the given interval it has only one maximum amounting to

$$f_{\max} = 2/\left(3\sqrt{3}\right).$$

When the maximal value is inserted into the inequality (17), the following is obtained

$$8 a \pi /\left(3\sqrt{3}\right) < 1 \;.$$

Thus, in effect

$$a < 3\sqrt{3}/(8\pi) \cong 0{,}2067 \;. \qquad (19)$$

Eventually, we have the following interval of admissible values of the shape parameter

$$0 \leq a < a_{gr} = 3\sqrt{3}/(8\pi)\;. \qquad (20)$$

## III. POTENTIAL ENERGY OF THE SYSTEM

Let us consider for the functional (11) the deflection function described by the Equation (14). After substituting this function in the Equation (11) *J*[*y*] becomes a function of a single variable *A*.

$$J[y] = V(A).$$

To find the value of *A* coefficient, let Ritz method be applied [19]. The method uses the necessary condition of existence of *V*(*A*) extreme, that is the equation

$$dV/dA = 0 \;. \qquad (21)$$

The Equation (11) must be first transformed and individual integrals calculated.
For the first addend it follows that

$$q\int_0^l y\,ds = q\int_0^l \frac{4}{3}A\sin^3\left(\frac{\pi s}{l}\right)ds = \frac{16\,q\,l}{9\pi}A\,. \quad (22)$$

In the second addend of the Equation (11) there is the $x_B$, value which is the $x$ coordinate of the movable end of elastica. It is calculated in the following way. Using the formula $ds^2 = dx^2 + dy^2$, we obtain

$$x_B = \int_0^l \sqrt{1-\left(\frac{dy}{ds}\right)^2}\,ds\,.$$

Since this integral cannot be calculated precisely, the approximate square root formula must be used here, leading to the result

$$x_B \cong \int_0^l \left[1-\frac{1}{2}\left(\frac{dy}{ds}\right)^2\right]ds = \int_0^l ds - \frac{1}{2}\int_0^l \left(\frac{dy}{ds}\right)^2 = l - \frac{1}{2}\int_0^l \left[\frac{4A\pi}{l}\sin^2\left(\frac{\pi s}{l}\right)\cos\left(\frac{\pi s}{l}\right)\right]^2 ds\,.$$

After transformation, we obtain

$$x_B = l - \frac{\pi^2 A^2}{2l}\,. \quad (23)$$

Thus, definitely

$$P\,x_B = P\,l - \frac{P\pi^2 A^2}{2l}\,. \quad (24)$$

Before calculation of the third addend, it is necessary to represent the curvature $d\varphi/ds$ in a somewhat different form. It is known that

$$\frac{dy}{ds} = \sin\varphi\,,$$

$$\frac{d^2 y}{ds^2} = \cos\varphi\,\frac{d\varphi}{ds}\,.$$

Thus

$$\frac{d\varphi}{ds} = \frac{d^2 y/ds^2}{\sqrt{1-(dy/ds)^2}}\,. \quad (25)$$

For the third addend it follows that

$$\frac{1}{2}EI\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds\,.$$

Now the integral has to be calculated.

$$\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds = \int_0^l \left(\frac{d^2y/ds^2}{\sqrt{1-(dy/ds)^2}}\right)^2 ds = \int_0^l \frac{\left(d^2y/ds^2\right)^2}{1-(dy/ds)^2} ds .$$

These type integrals are discussed among others in publications [20] and [18]. As it is impossible to represent the result of the above integration in the form of elementary functions, an approximate solution is to be applied. To do this, the numerator and denominator of the integral are multiplied by $1+(dy/ds)^2$ .
Then, the product is

$$\int_0^l \frac{\left(d^2y/ds^2\right)^2}{1-(dy/ds)^2}\frac{1+(dy/ds)^2}{1+(dy/ds)^2} ds = \int_0^l \frac{\left(d^2y/ds^2\right)^2\left[1+(dy/ds)^2\right]}{1-(dy/ds)^4} ds . \quad (26)$$

Since, as it was shown at the beginning: $dy/ds < 1$, then $(dy/ds)^4$ is much less than 1. It can be thus assumed that $1-(dy/ds)^4 \cong 1$, and thus it follows

$$\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds \cong \int_0^l \left(d^2y/ds^2\right)^2\left[1+(dy/ds)^2\right] ds . \quad (27)$$

A better approximation can be obtained by subsequent multiplication of the numerator and denominator of the Equation (26) by $1+(dy/ds)^4$ and so on. Applying in Equation (27) the formulae for the first and second derivatives of the *y* function, it is obtained after integration.

$$\frac{1}{2} EI \int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds = \frac{EI\pi^4}{8l^5}\left(20\,l^2A^2 + 13\,\pi^2A^4\right). \quad (28)$$

Eventually, the formula for total potential energy of the system takes the form of

$$V(A) = Pl - \frac{P\pi^2}{2l}A^2 + \frac{16\,ql}{9\pi}A + \frac{5EI\pi^4}{2l^3}A^2 + \frac{13\,EI\pi^6}{8l^5}A^4 . \quad (29)$$

The potential energy is so the function of the single variable *A* which can be called a variable parameter of shape, and two constants connected with the external load, namely *P* and *q*.

## IV. ANALYSIS OF STATES OF EQUILIBRIUM

To begin analysis of the states of equilibrium, the above-mentioned condition (21) is to be applied, on basis of which the value of the shape parameter *A* can be determined. It is the parameter on that the kind of equilibrium depends with a given load defined by *P* and *q*.

Thus, we have

$$\frac{dV}{dA} = -\frac{P\pi^2}{l}A + \frac{5EI\pi^4}{l^3}A + \frac{13\,EI\pi^6}{2l^5}A^3 + \frac{16\,ql}{9\pi} = 0 . \quad (30)$$

From the Equation (30) the following relationship between the force *P* and parameter *A* is calculated

$$P = \frac{5EI\pi^2}{l^2} + \frac{13EI\pi^4}{2l^4}A^2 + \frac{16ql^2}{9\pi^3}\frac{1}{A}. \quad (31)$$

To make further discussion more general, let us represent the energy *V*, force *P* and continuous load *q* in the dimensionless form, relating them to Euler critical force $P_{cr} = \pi^2 EI / l^2$.

Let us make the following transformations

$$a = \frac{A}{l}, \qquad p = \frac{P}{P_{cr}},$$

$$(32)$$

$$w = \frac{ql}{P_{cr}}, \qquad v = \frac{V}{P_{cr}l}.$$

Thus, we get

$$v = p - \frac{\pi^2}{2}pa^2 + \frac{5\pi^2}{2}a^2 + \frac{13\pi^4}{8}a^4 + \frac{16w}{9\pi}a, \quad (33)$$

$$p = 5 + \frac{13\pi^2}{2}a^2 + \frac{16w}{9\pi^3}\frac{1}{a}. \quad (34)$$

Since the relation (34) was obtained by use of the Equation (21) expressing the necessary condition for existence of the function extremum, thus points lying on the *p* curves correspond to extreme of the function of potential energy.

Location of the minimum and maximum of energy must still be defined. Here, the second derivative of potential energy is used as equated to zero

$$\frac{d^2V}{dA^2} = -\frac{P\pi^2}{l} + \frac{5EI\pi^4}{l^3} + \frac{39EI\pi^6}{2l^5}A^2 = 0.$$

Therefore

$$g = \frac{P}{P_{cr}} = \frac{39\pi^2}{2}a^2 + 5. \quad (35)$$

The curve *g* described by the Equation (35) is a diagram of the compressing force *p* represented as function of the parameter *a*, but corresponding only to the points for which the second derivative $d^2v/da^2$ (or in another way $d^2V/dA^2$) is equal to zero.

Figure 4 presents dependence of the force *p* on the dimensionless parameter *a* for several values *w*, and the drawn curve *g*. This is a boundary curve. Right from it, on each of the *p* curves, with *w*>0 there are points for which $d^2v/da^2 > 0$, which corresponds to the minimum of potential energy *v*, that is to the state of stable equilibrium. It should be noted moreover that the curve *g* crosses the functions *p* in their minimal points. The boundary value of the shape parameter $a_{gr}$ is also marked in the diagram.

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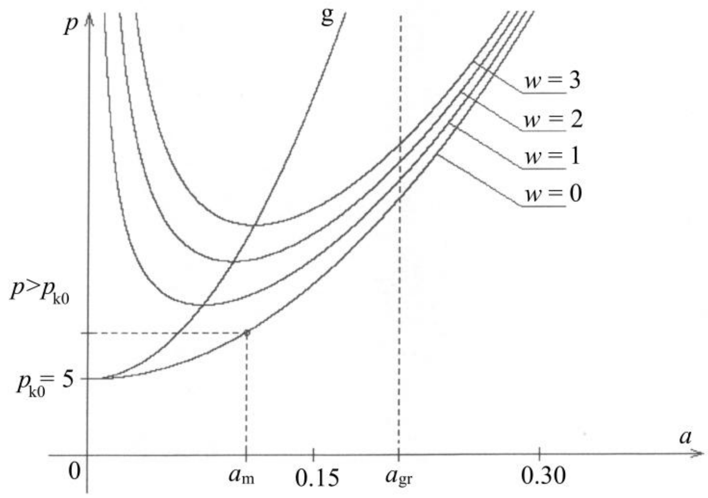


**Figure 4:** A diagram of the compressing force $p$ corresponding to the extreme of potential energy.

Now let us discuss in more detail the states of equilibrium for two possible cases of continuous load $w$ ($w$=0 and $w$>0). It should be remembered that everything is considered with the condition $A$>0 or, which follows $a$>0.

Case I ($w$=0).

Here are considered Figure 4 and the formulae of potential energy $v$ and its derivatives. Substituting $w$=0 in the Equation (33) we have

$$v = p - \frac{\pi^2}{2} p a^2 + \frac{5\pi^2}{2} a^2 + \frac{13\pi^4}{8} a^4 ,$$

$$\frac{dv}{da} = -\pi^2 p a + 5\pi^2 a + \frac{13\pi^4}{2} a^3 ,$$

$$\frac{d^2 v}{da^2} = -\pi^2 p + 5\pi^2 + \frac{39\pi^4}{2} a^2 .$$

It can be noticed that minimal value of the force $p$ amounts to $p_{k0}$=5 and it occurs with $a$=0.

- If $p<p_{k0}$, then the function $v$ has the only extremum for $a$=0 and it is its minimum ($dv/da = 0$, $d^2v/da^2 > 0$ for $a$=0).
- If $p=p_{k0}$, then the function $v$ also has its minimum for $a$=0, but it is more flat in this point (for $a$=0 all differential coefficients of the function $v$ with respect to $a$ up to the third degree inclusive are equal to zero, while $d^4v/da^4 > 0$ ).
- In case when $p>p_{k0}$, then the derivative $dv/da$ when $a$>0 has already two zero points: one for $a$=0, the other for $a=a_m$ defined by the equation

$$a_m = (1/\pi)\sqrt{2(p-5)/13} . \quad (36)$$

It can be checked that for $a$=0 the derivative $d^2v/da^2 < 0$, so in this point there is the maximum of potential energy $v$. On the other hand, for $a=a_m$ the derivative $d^2v/da^2 > 0$, so there is the minimum of potential energy $v$.

To conclude, for $p \le p_{k0}$ there exists only the rectilinear form of equilibrium, that is the stable position is only for $a$=0.

However if $p>p_{k0}$, there are two positions of equilibrium. First one, for $a$=0 is unstable, whereas the other, for $a_m$ defined by the Equation (36) is the position of stable equilibrium.

Case II ($w>0$)

Like above, here are considered the formulae of potential energy $v$ and its derivatives.

$$v = p - \frac{\pi^2}{2} p a^2 + \frac{5\pi^2}{2} a^2 + \frac{13\pi^4}{8} a^4 + \frac{16 w}{9\pi} a ,$$

$$\frac{dv}{da} = -\pi^2 p a + 5\pi^2 a + \frac{13\pi^4}{2} a^3 + \frac{16 w}{9\pi} ,$$

$$\frac{d^2 v}{da^2} = -\pi^2 p + 5\pi^2 + \frac{39\pi^4}{2} a^2 .$$

Basing on Figure 4 it can be seen that the minimal value of the force $p$, which for further consideration will be marked as $p_k$, is greater than it was in the previous situation for $w=0$ ($p_k>p_{k0}=5$).

It can be also seen that the minimum occurs with $a>0$. Let this point be designated as $a_k$ like in Figure 5. To determine the values $p_k$ and $a_k$ the minimum of the function $p$ given by the Equation (34) must be found. After appropriate transformation, we obtain

$$p_k = 5 + \sqrt[3]{416\ w^2 / (3\pi^4)} , \quad (37)$$

$$a_k = \sqrt[3]{16\, w / (117\ \pi^5)} . \quad (38)$$

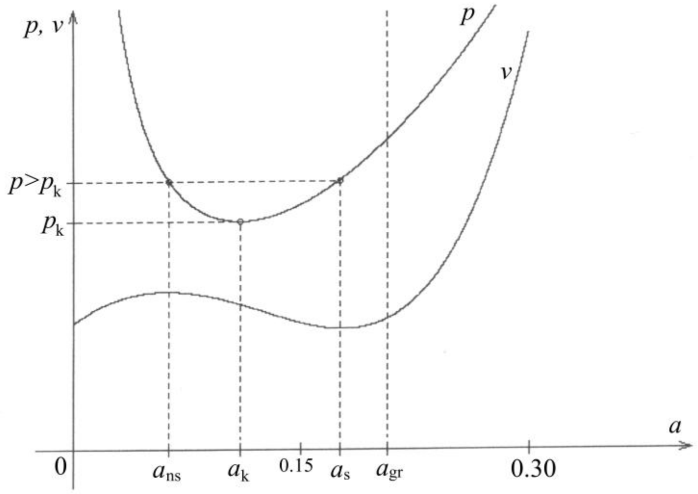


**Figure 5:** An example of a diagram of the compressing force $p$ for $w=3$ and potential energy $v$ for $w=3$ and $p=8$.

- If $p<p_k$, then the function $v$ in the interval for $a\geq0$ has no extremum. In the point $a=0$ and for each $a>0$ the derivative $dv/da > 0$, so the function $v$ is increasing while the value $a$ increases. From the function analysis it follows that for $a=0$ the potential energy accepts the least value in the present interval.
- If $p=p_k$, then for $0<a<a_k$ the derivative $dv/da > 0$, so the function $v$ is increasing. In the point $a=a_k$ derivatives of the function $v$ up to the second degree inclusive with respect to $a$, are equal to zero, while $d^3v/da^3 > 0$ which means that in this point there is the point of inflexion. For $a>a_k$ it again $dv/da > 0$, so in this interval the function $v$ is increasing again.
- The situation of $p>p_k$ is illustrated as example in Figure 5 together with the diagrams of the force $p$ for $w=3$ and of the potential energy $v$ in case when $w=3$ and $p=8>p_k$ (for our example $p_k$=7,3399 and $a_k$=0,1103).
  In point $a=0$ the derivative $dv/da > 0$. Based on Figure 5 it can be clearly seen that while increasing the value $a$ from point $a=0$ the function $v$ is increasing up to the local maximum which is attained at $a=a_{ns}$. Then the function is decreasing till the local minimum occurring at $a=a_s$, and increases again.

To conclude, if $p<p_k$, then there is only a rectilinear form of stable equilibrium (state of stability for $a=0$). If $p=p_k$, then the stable equilibrium occurs also for the point $a=0$ while at the point $a=a_k$ there is the critical state in which the neutral equilibrium occurs (the deflection point in the diagram of energy $v$).
Eventually, for $p>p_k$ there are three forms of equilibrium.
1. Rectilinear form for $a=0$ corresponds with the state of stable equilibrium.
2. Curvilinear form corresponding to the left part of the curve (for $a=a_{ns}$) is unstable (local maximum of energy $v$).
3. Curvilinear form corresponding to the right part of the curve (for $a=a_s$) is stable (local minimum of energy $v$).

Let us now calculate the points $a_s$ and $a_{ns}$.
Considering that $a\geq 0$ let multiply both sides of the Equation (34) by $a$. We obtain then

$$\frac{13\pi^2}{2}a^3+(5-p)a+\frac{16w}{9\pi^3}=0\,. \quad (39)$$

The Equation (39) is a cubic equation with respect to $a$.
Roots of $a$ have to be calculated with the given $p$ and $w$. According to the earlier analysis, for $p>p_k$ and under assumption that $a\geq 0$ there should be two roots, respectively of $a_s$ and $a_{ns}$, while $a_{ns}<a_s$.
To solve the Equation (39) Cardan's formulae will be applied.
The Equation (39) can be transformed to the shape of

$$a^3+a(10-2p)/(13\pi^2)+(32w)/(117\pi^5)=0\,. \quad (40)$$

The roots of Equation (40) depend on the value of the expression

$$R=(1/4)\left[32w/(117\pi^5)\right]^2+(1/27)\left[(10-2p)/(13\pi^2)\right]^3\,. \qquad (41)$$

If $p>p_k$, then $R<0$. It means that the Equation (40) has three real roots. They are

$$a_{1,2,3}=2\sqrt{\frac{2p-10}{39\pi^2}}\cos\left(\frac{\lambda}{3}+k\frac{2\pi}{3}\right),\quad (k=0,1,2)\,. \qquad (42)$$

The angle $\lambda$ occurring in the Equation (42) is calculated from the Equation

$$\cos(\lambda)=-\sqrt{\frac{416w^2}{3\pi^4(p-5)^3}}\,. \quad (43)$$

From the analysis of the function $\cos(\lambda)$ it follows that for $p=p_k$ $\cos(\lambda)=-1$ that is $\lambda=\pi$, while for $p\to\infty$, $\cos(\lambda)\to 0$, so $\lambda\to\pi/2$. The root of $a_2$ of the Equation (42) for $k=1$ is always negative, so must be rejected. There are two roots left, of which $a_1$ (for $k=0$) is greater than $a_3$ (for $k=2$).
Eventually we have that

$$a_s=2\sqrt{\frac{2p-10}{39\pi^2}}\cos\left(\frac{\lambda}{3}\right), \quad (44)$$

$$a_{ns}=2\sqrt{\frac{2p-10}{39\pi^2}}\cos\left(\frac{\lambda}{3}+\frac{4\pi}{3}\right), \qquad (45)$$

where the angle $\lambda$ is defined by the Equation (43).

For example from Figure 5 we obtain for $w$=3 and $p$=8 the values $a_s$=0,1781, $a_{ns}$=0,0626.

## V. DISCUSSION OF THE RANGE OF VALUES OF AXIAL FORCE AND CONTINUOUS LOAD

In point 3 it was stated that $a$ cannot take any optional value due to the specific length of the elastica. Admissible values of $a$ belong to the interval: $0 \le a \le a_{gr}$ where $a_{gr} = 3\sqrt{3}/(8\pi) \cong 0{,}2067$.

On analyzing Figure 4 it is seen that if we increase the value $w$, then the stable curvilinear solutions $a_s$ located right from the curve $g$, will be for the value $w$ above a certain amount greater than $a_{gr}$ .

Since we try to be always within the admissible limits of the value $a$, then let us consider what the maximum value $w$ should be for stable curvilinear solutions $a_s$ to belong still to the interval.

Let the value $a_k$ defined by the Equation (38) be less than $a_{gr}$ . By virtue of the above

$$\sqrt[3]{16\,w/\left(117\,\pi^5\right)} < 3\sqrt{3}/(8\pi)\ ,$$

thus

$$w < 9477\sqrt{3}\,\pi^2/8192 = w_{max}\ . \quad (46)$$

Approximately, the boundary value for $w$ amounts to $w_{max} \cong 19{,}7761$ .

Similarly, when looking at Figure 5 it can be seen that with fixed $w$ increasing of the force $p$ above a certain value results in the value $a_s$ greater than the admissible one.

Since the value $a_s$ depends not only on the value $p$, but also on the value $\lambda$ (dependent in turn on $w$), thus for various $w$ the maximal values of the axial force $p_{max}$ will be different, and above them there are no more stable curvilinear solutions in the discussed interval of admissible values of $a$.

In case when $w$=0 it is sufficient for the value $a_m$ as defined by the Equation (36) was less than $a_{gr}$ . Thus

$$\frac{1}{\pi}\sqrt{\frac{2(p-5)}{13}} < \frac{3\sqrt{3}}{8\pi}\ .$$

Therefore

$$p < 5 + \frac{351}{128} = p^0_{max}\ . \quad (47)$$

Approximate value of the maximal axial force in case of $w$=0 amounts to $p^0_{max} \cong 7{,}7422$ .

For the second case of $w$>0, the values of $p_{max}$ for subsequent $w \le w_{max}$ are calculated numerically.

To this end, with a fixed $w$, the value $a_s$ was calculated from the Equation (44) for subsequent forces $p$ increasing by even steps, beginning from the value $p_k$, till the moment of exceeding the value $a_{gr}$ .

Then, the calculation was repeated for the next value of $w$. From the obtained values, a diagram of maximal axial force $p_{max}$ as function of continuous load $w$ was drawn up.

Basing on the Equation (37), also a diagram of the critical force $p_k$ as function of continuous load $w$ was made. Both the diagrams are presented in Figure 6.

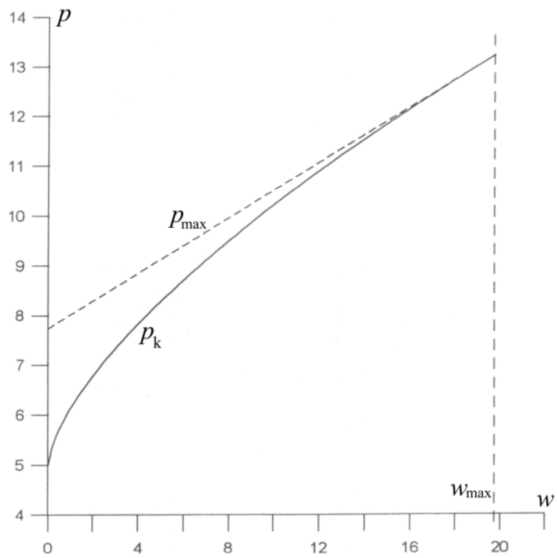


**Figure 6:** A diagram of the maximal force $p_{max}$ and critical force $p_k$ represented as function of the continuous load *w*.

## VI. CONCLUSIONS

In conclusion let us sum up the problem of stability of the discussed elastica in case when $w>0$. As it follows from the above discussion, in this case the rectilinear form of equilibrium will be always stable, while we consider infinitesimal deviations from the point of balance. It can be seen clearly that with the increasing value *p* the local maximum of energy occurs at a approaching more and more $a=0$, but not reaching it. It evidences only the fact that for great *p* it is easier to unbalance the system when it is in stable, rectilinear state of equilibrium, causing some finite displacement to it. In order, however, that the system assumes a new, curvilinear form of stable equilibrium, it is necessary to pass the maximum of potential energy corresponding to the unstable form of equilibrium (Figure 5). The greater the force *p* is, the less displacement is needed for the system to assume a new form of equilibrium.

The force $p_k$ as defined by the Equation (37) can be called the critical force, above which except for the rectilinear form of stability there is also a curvilinear form of stable equilibrium of the system.
Due to the assumption of inextensibility of the elastica there arose the limitation of the value of the shape parameter *a*, which has to be less than $a_{gr}$ . Thus, in turn, there followed some limitations for the value *w* and axial force *p* in the determination of the curvilinear stable solutions $a_s$ in the limits of admissible values *a*, which are illustrated in Figure 6.

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