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Research Paper

Invariants and constraints of a hollow hyperelastic isotropic cylindrical tube with a deformed radius or not

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***Abstract**:*
In this research paper we have proposed as a work, the simulation of elementary invariants, components of the Cauchy stress tensor and internal pressure in the case of a cylindrical hyperelastic tube undeformed or deformed by application to a constant, decreasing and increasing radius. To illustrate the radial deformation of the hyperelastic cylindrical tube, we used a perturbation parameter in the kinematics with an isotropic and compressible strain energy function in the Cauchy stress tensor to obtain our desired expressions. The comparison of the simulated expressions allowed us to see differences and equalities in the elementary invariants, resemblances in the components of the Cauchy stress tensor and the variation of the pressure in each case, ie for a increasing and decreasing radius. Finally, these simulations have allowed us to prove how these disturbances can perturb invariants and constraints in a hollow hyperelastic isotropic cylindrical tube.
Keywords: *isotropic, compressible, incompressible, elementary invariants, Cauchy stress tensor, internal pressure, disturbance parameter, decreasing and increasing radius.*

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I. Introduction

The description of the anisotropic hyperelastic mechanical behavior of a mechanical cylindrical tube is still useful to better understand the diseases that plague the cardiovascular system [1].To achieve these studies many variables and tensors must be considered, the deformation gradient tensor F which is nothing more than the tangent linear application serves to describe the state of local deformation resulting from the internal forces. In the biomechanical modeling, there are various strain energy functions which allow to realize such work among which we can quoted the polynomial, exponential, power or logarithmic form [2]. These energy potentials have been established as part of a phenomenological approach that describes the macroscopic nature of the material and there are functions of elementary invariants [3]. Most of these mechanical studies have different and diverse objectives, one part of this is most often concentrated in the analysis of stresses and pressure in incopressible, in the isotropic or anisotropic case [4,5,6]. In an other part there are interested to establish a direct correlation between the mechanisms of deformation and the physical characters of the structure and certain mathematical criterias which are known, such as convexity, ellipticity and objectivity should normally be satisfied by the strain energy functions [7,8]. As a contribution in the biomechanical modeling, we study a smooth hollow cylindrical structure or with an internal deformation in the compressible or incompressible isotropic case by application to a decreasng and increasing radius. The elementary invariants, the non-zero components of the Cauchy stress tensor and the internal pressure will be simulated and analyzed in the case of internal radius deformation in order to highlight the differences that can be observed between these parameters which are studied according to whether that the radius decreases or increases what can represent respectively in the reality a pathological artery.

II. Preliminaries

A continuous material body in which a material particule is described by $\mathbf{X}$ in hte undeformed reference configuration and by $\mathbf{x}$ in the deformed configuration.The local deformation is described by the deformation gradient tensor $\mathbf{F}$ which is the tangent linear application. This tensor allow to have the volume change, his determinant is always positive, so that

$$\mathbf{x} = \chi(\mathbf{X}); \quad \mathbf{F} = Grad\,\mathbf{x}; \quad det\,\mathbf{F} \geq 0. \tag{1}$$

In incompressible deformation, there is no volume change so the determinant is equal to 1. The deformation gradient admits polar decompositions into rotation $\mathbf{R}$ and right stretch $\mathbf{U}$ or left stretch $\mathbf{V}$ with

$$\mathbf{U} = \mathbf{F}\mathbf{R}^{-1}; \quad \mathbf{V} = \mathbf{R}^{-1}\mathbf{F}. \tag{2}$$

Where $\boldsymbol{R^{-1}}$ is the reverse matrix of $\mathbf{R}$. It is important to specify from the following tensors $\mathbf{U}$ and $\mathbf{V}$ that their square gives us familiar tensors that are: right Cauchy-Green tensor $\mathbf{C}$ and left Cauchy-Green tensor $\mathbf{B}$

$$\mathbf{U^2} = \mathbf{F^tF} = \mathbf{C}; \quad \mathbf{V^2} = \mathbf{FF^T} = \mathbf{B}. \tag{3}$$

In three dimension, these tensors admit representations in terms of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and the right and left principal stretch vectors which are respectively $(\mathbf{u^1}, \mathbf{u^2}, \mathbf{u^3})$ and $(\mathbf{v^1}, \mathbf{v^2}, \mathbf{v^3})$ [9].
However during his deformation, the mechanical behavior of a material is described by a thermodynamic potential W called strain energy function and is a function of the deformation gradient $\mathbf{F}$ and so it becomes a function of $\mathbf{C}$ or $\mathbf{B}$:

$$W = W(\mathbf{F}) = W(\mathbf{C}) = W(\mathbf{B}). \tag{4}$$

We note here that the tensor C makes it possible to measure the deformation in Lagrangian configuration and the tensor B makes it possible to measure

this same deformation in Eulerian configuration. These two configurations are equivalent and remain confused in the case of an infinitesimal deformation.
In our study we will limit ourselves to isotropic materials ie in the absence of fibers. So the energy W becomes a function of elementary invariants I_1, I_2 or I_1, I_2, I_3, what yields in compressible:

$$W = W(I_1, I_2, I_3); \tag{5}$$

and in incompressible:

$$W = W(I_1, I_2). \tag{6}$$

The Cauchy stress tensor $\mathbf{T}$ is nothing more than the response of the material to the constraints in which it is submit and derived from the strain energy function, so in the case of an incompressible and isotropic material, we have:

$$\mathbf{T} = -p\mathbf{1} + \left(\frac{\partial W}{\partial \mathbf{F}}\right)\mathbf{F^T} = -p\mathbf{1} + 2\left(\frac{\partial W}{\partial I_1}\right)\mathbf{B} - 2\left(\frac{\partial W}{\partial I_2}\right)\mathbf{B^{-1}}. \tag{7}$$

Here, the arbitrary parameter p represents the internal pressure.
In an other hand if we restrict our study to a hollow cylindrical tube, the stress state of this tube is represented by the Cauchy stress Tensor in [3] defined by:

$$\mathbf{T} = -\beta_0\mathbf{1} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1}; \tag{8}$$

where $\beta_i (i = 1, 2; 3)$ are function of elementary isotropic invariants so that:

$$\begin{aligned} \beta_0 &= 2I_3^{-1/2}[I_2W_2 + I_3W_3]; \\ \beta_1 &= 2I_3^{-1/2}W_1; \\ \beta_{-1} &= -2I_3^{1/2}W_2; \end{aligned} \tag{9}$$

with $W_i = \partial W/\partial I_i, (i = 1, 2, 3)$.
The condition of incompressibility means that $I_3 = 1$, so system (9) becomes

$$\begin{aligned} \beta_0 &= 2I_2W_2; \\ \beta_1 &= 2W_1; \\ \beta_{-1} &= -2W_2. \end{aligned} \quad (10)$$

By identification between (7) and (10),we obtain

$$p = 2I_2W_2. \quad (11)$$

The equation of equilibrium with no body force are given in [10] by

$$div \ \ \mathbf{T} = 0. \quad (12)$$

let's consider a continuous cylindrical hyperelastic tube where a material point occupies the position (R, Θ, Z) before the deformation and the position (r, θ, z) after deformation in [11] and which is represented by the following kinematic

$$r = r(R); \quad \theta = \Theta; \quad z = \lambda Z. \quad (13)$$

here λ represente the elongation of the tube.
The gradient tensor of deformation is difined by:

$$\mathbf{F} = \begin{pmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}; \quad (14)$$

where $\lambda_r = \partial r/\partial R$,$\lambda_\theta = \partial\theta/\partial\Theta$ and $\lambda_z = \lambda$.
It's follows the symmetrical right and left Cauchy-Green tensors which are

$$\mathbf{C} = \mathbf{B} = \begin{pmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix}. \quad (15)$$

We can then calculate the first three isotropic elementary invariants of deformation:

$$\begin{aligned} I_1 &= tr(\mathbf{C}) = tr(\mathbf{B}) = \lambda_r^2 + \lambda_\theta^2 + \lambda_z^2; \\ I_3 &= det(\mathbf{C}) = \lambda_r^2\lambda_\theta^2\lambda_z^2; \\ I_2 &= I_3\mathbf{C}^{-1} = I_3\mathbf{B}^{-1} = \lambda_r^2\lambda_\theta^2 + \lambda_\theta^2\lambda_z^2 + \lambda_z^2\lambda_r^2. \end{aligned} \quad (16)$$

In the absence of volume forces, the no null components of the Cauchy stress tensor defined in (8) with the condition of incompressibility are:

$$\begin{aligned} T_{rr} &= 2\left[-\left(\lambda_r^2\lambda_\theta^2 + \lambda_\theta^2\lambda_z^2 + \lambda_z^2\lambda_r^2\right)W_2 + \lambda_r^2W_1 + \lambda_r^{-2}W_2\right]; \\ T_{\theta\theta} &= 2\left[-\left(\lambda_r^2\lambda_\theta^2 + \lambda_\theta^2\lambda_z^2 + \lambda_z^2\lambda_r^2\right)W_2 + \lambda_\theta^2W_1 + \lambda_\theta^{-2}W_2\right]; \end{aligned} \quad (17)$$

$$T_{zz} = 2\left[-\left(\lambda_r^2\lambda_\theta^2 + \lambda_\theta^2\lambda_z^2 + \lambda_z^2\lambda_r^2\right)W_2 + \lambda_z^2W_1 + \lambda_z^{-2}W_2\right].$$

And the relation (11) becomes

$$p = \left(\lambda_r^2\lambda_\theta^2 + \lambda_\theta^2\lambda_z^2 + \lambda_z^2\lambda_r^2\right)W_2. \quad (18)$$

III. Application to an artery affected by stenosis

In this part, we consider an artery affected by stenosis.

IV. Application to an undeformed or deformed hyperelastic cylindrical tube

In this paragraph we will make an application on two cases. In the first part we will consider a normal hyperelastic cylindrical tube, ie with a constant radius and secondly a deformed hyperelastic cylindrical tube, ie with a variable radius. We will focus on the study of elementary invariants, non-zero components of the Cauchy stress tensor and the internal pressure.

4.1 Case of an undeformed hyperelastic cylindrical tube

Let's consider a material supposed to be a continuous hyperelastic cylindrical tube which is represented by the following kinematic

$$r = R, \quad \theta = \Theta, \quad z = Z. \tag{19}$$

The gradient tensor of deformation becomes equivalent to the identity tensor:

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

It's follows the right and left Cauchy-Green tensors which are equivalent to the gradient tensor

$$\mathbf{C} = \mathbf{B} = \mathbf{F}. \tag{21}$$

We can then calculate the three isotropic elementary invariants of deformation by others expressions in [12]:

$$\begin{aligned} I_1 &= tr(\mathbf{C}) = tr(\mathbf{B}) = 3; \\ I_3 &= det(\mathbf{C}) = det(\mathbf{C}) = 1; \\ I_2 &= tr(\mathbf{C}^*) = tr(\mathbf{B}^*) = 3. \end{aligned} \tag{22}$$

In the absence of volume forces, the no null components of the Cauchy stress tensor defined in (17) become all null because of the invariants which are constant:

$$T_{rr} = 0, \quad T_{\theta\theta} = 0, \quad T_{zz} = 0. \tag{23}$$

And so no extra pressure ($p = 0$).

Remark:
For an underformed hyperelactic cylindrical tube, we find that the fact that the elementary invariants are constant renders null all the components of the Cauchy stress tensor and the extra pressure.

4.2 Case of a deformed hyperelastic cylindrical tube

Here we consider a material supposed to be a continuous hyperelastic cylindrical tube, in which we have a radial deformation illustrated with a disturbance parameter ε [13] by the following kinematic

$$r = \varepsilon R, \quad \theta = \Theta, \quad z = Z. \tag{24}$$

where ε which is positive as $\varepsilon < 1$ or $\varepsilon > 1$ depending to case of decreasing or increasing radius. This variation of the radius implies the variation of the circumference, so the local variation of volume which automatically places us in

a situation of compressibility for the rest of the study.
The gradient tensor of deformation becomes:

$$\mathbf{F} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

We can see from here that the rest of our study becomes independent of the initial radius R, the only remaining variable is the disturbance parameter ε.
It's follows the right and left Cauchy-Green tensors given by:

$$\mathbf{C} = \mathbf{B} = \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

From relation (26), the first three isotropic elementary invariants of deformation become:

$$\begin{aligned} I_1 &= tr(\mathbf{C}) = tr(\mathbf{B}) = 2\varepsilon^2 + 1; \\ I_3 &= det(\mathbf{C}) = det(\mathbf{C}) = \varepsilon^4; \\ I_2 &= I_3\mathbf{C}^{-1} = I_3\mathbf{B}^{-1} = \varepsilon^2(\varepsilon^2 + 2). \end{aligned} \tag{27}$$

In the compressible case, the relations (27) in (9) gives:

$$\begin{aligned} \beta_0 &= 2\left[\left(\varepsilon^2 + 2\right) W_2 + \varepsilon^2 W_3\right]; \\ \beta_1 &= 2\varepsilon^{-2} W_1; \\ \beta_{-1} &= -2\varepsilon^2 W_2. \end{aligned} \tag{28}$$

The absence of volume forces yields the non null components of the Cauchy stress tensor:

$$T_{rr} = T_{\theta\theta} = T_{zz} = 2\left[W_1 - \left(\varepsilon^2 + 3\right) W_2 - \varepsilon^2 W_3\right]. \tag{29}$$

For a reason to be able to simulate our biological environment in order to better understand what is happening in the reality, let's consider the strain energy function of Diouf-Zidi [14] defined by:

$$W = W_{n=2} = \frac{\mu}{2}\left[(I_1 - 3) + a_1 (I_2 - 3) + a_2 \left(I_3^{1/2} - 1\right)^2\right]. \tag{30}$$

This previous relation (30) allows us to calculate expressions of W_i, which gives:

$$W_1 = \frac{\mu}{2}; \quad W_2 = \frac{\mu a_1}{2}; \quad W_3 = \frac{\mu a_2}{2}\left(1 - \varepsilon^{-2}\right). \tag{31}$$

Using expressions (31) in (30), we find:

$$T_{rr} = T_{\theta\theta} = T_{zz} = \mu\left[1 - a_1\left(\varepsilon^2 + 1\right) - a_2\left(\varepsilon^2 - 1\right)\right]. \tag{32}$$

And with all the hypothesis enumerate in this section the pressure expression becomes:

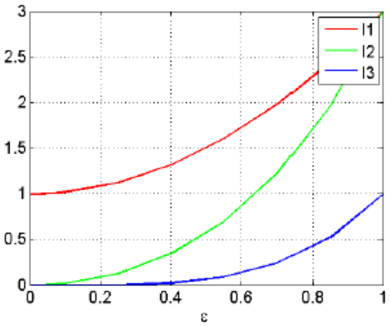
$$p = \mu\left[a_1\left(\varepsilon^2 + 2\right) + a_2\left(\varepsilon^2 - 1\right)\right]. \tag{33}$$

4.2.1 With decreasing radius

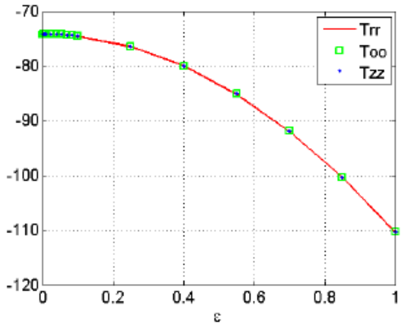
Here we consider a progressive decreasing radius, ie the relation (24) with an $\varepsilon \in [0; 1]$ and the following material parameters in [15]

$\mu(kPa)$	$a_1(kPa)$	$a_2(kPa)$
3.0544	1.45	0.1239

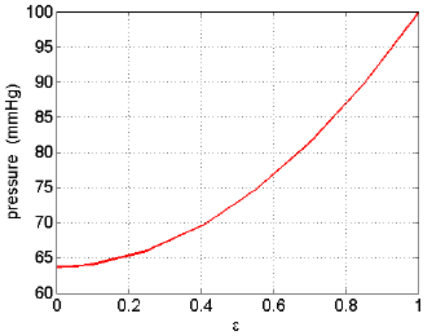
A simulation of elementary invariants, non-zero components of the Cauchy stress tensor in $(mmHg)$ and the pressure with these given parameters yields the following graphs



The elementary invariants



The Cauchy stress tensor components



The three previous figures show us that in the case of progressive decreasing radius:
-The elementary invariants decrease with I_1 larger than I_2 and I_2 larger than I_3. At the beginning of the radius decrease, we have I_1 and I_2 which are close to each other and more the radius progresses in decreasing, more I_2 decreases faster than I_1 and when ε is close to zero, we can clearly see that I_2 is getting closer to I_3 with an average variation of local volume. All the invariants show an exponential decreasing with I_2 which has a faster decreasing progresse between the maximum value of I_1 and the minimum value of I_3.

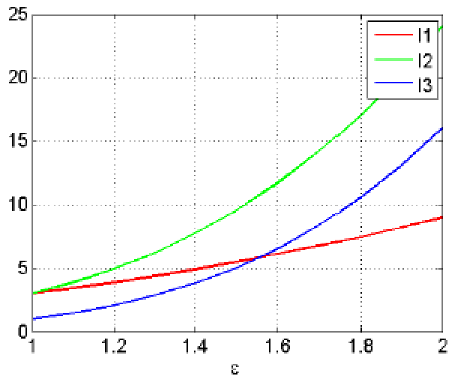
-The three non-zero components of the Cauchy stress tensor remain identical and negative and it decrease exponentially. The negative components show us that the tube remains compressed when the radius decreases.
-The pressure gradually decreases following an exponential pace when the tube is progressively invaded by the radius decrease.

4.2.2 With increasing radius

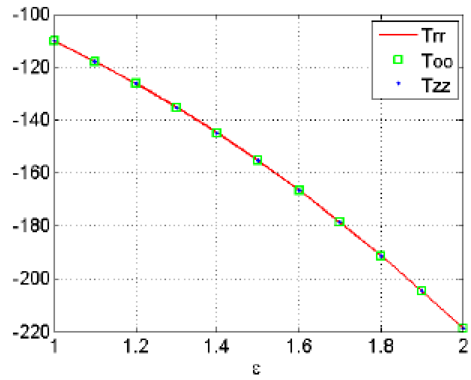
In this subsection we consider a progressive increasing radius, ie the relation (24) with an $\varepsilon \in [1;2]$ this same material parameters in [15]

$\mu(kPa)$	$a_1(kPa)$	$a_2(kPa)$
3.0544	1.4539	0.1239

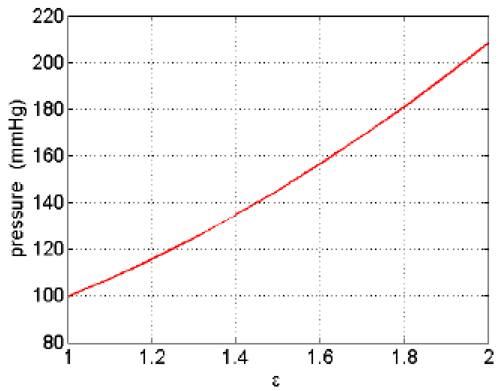
The simulation of elementary invariants, non-zero components of the Cauchy stress tensor in $(mmHg)$ and the pressure with these following given parameters yields this following graphs



The elementary invariants



The Cauchy stress tensor components



These figures show us that in the case of progressive increasing radius:
-We have the elementary invariants which increases with I2 larger than I1 and I3. At the beginning of the radius increase, we have I1 which is greater than I3 which remains valid until radius reaches a size exceeding 1.55 times its initial size. once this size exceeds, I1 becomes smaller than I3 with a large variation in local volume. It should be noted that I2 and I3 grow exponentially while I1 remains close to logarithmic growth.
-The three non-zero components of the Cauchy stress tensor remain also identical and negative. It decreases exponentially but faster than in the case of the decreasing radius. The negative components also show us that the tube is more compressed when the radius increases.
-The pressure increases exponentially when the radius progresse in decreasing.

V. Conclusion

In this paper we have proposed a modelization of the elementary invariants, components of the Cauchy stress tensor and the internal pressure in a hollow cylindrical tube of constant or variable radius. We used a disturbance parameter to illustrate the variations of the tube radius in our study. The Diouf-Zidi strain enrgy functions model which is a power type were used for the realization of this work. this study allowed us to highlight the similarities and differences that can be observed at the level of the isotropic elementary invariants, to see the resemblance at the level of the components of the Cauchy stress tensor, according to whether we are in compressible or incompressible deformation on the one hand, but also to show on the other hand that for each type of variation of the tube circumference, there is a dysfunction which in turn gives rise to other anormalities. We finally prove through simulations, how a tube malformation can cause perturbation within it and how these disturbances can be destructive if they are not stoped on time.

VI. Outlooks

As perpectives of our learning in biomechanic, a thorough study of cases of decreasing and increasing radius of the tubular hyperelastic tube can allow us to better understand how pathological artery perturb our cardiovascular system. -The result of decreasing radius can represent a stenosis artery in the reality. As a consequence of this imbalance, we attend to a compressed artery, a blood pressure that gradually decreases (low blood pressure), what will lead a malnitrution of the organs nourished by this artery and once the artery is obstruct, it follows a loss of sensitivity and then a death of this concerned organs. -The result of the increasing radius can represent an artery with aneurysm. As a consequence of these variations, we have a compressed artery, a blood pressure that gradually increases (high blood pressure), which end by a rupture of the artery, followed by an internal haemorrhage which can be fatal in certain cases.

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