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Research Paper

Numerical Solution of the Troesch Problem by a Bracketing Shooting Approach

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Abstract: *A shooting method utilizing a bracketing approach is utilized for the solution of the Troesch problem. Bracketing methods though slow to converge, are robust, easy to apply and are guaranteed to obtain a solution since the root is trapped within an interval. This is as far as the resolution of the nonlinearity inherent in the Troesch problem is concerned. However the major thrust of our approach lies in the numerical solution of the highly nonlinear Troesch problem by adopting a very simple and direct approach. We follow the path of converting by a variable transformation the hyperbolic nonlinearity in the problem to its polynomial analog. The resulting governing equation though more complicated is less stiff and easier to deal with. The numerical results obtained herein are not only in consonance with the physics of the problem, but also display excellent convergence.*

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I. Introduction

The Troesch nonlinear two-point boundary value problem (BVP) has been applied to study the theory of gas in porous electrodes as well as the confinement of a plasma column by radiation pressure [1,2,3]. It can be represented as:

$$y'' = \lambda \sinh(\lambda y) \qquad (1)$$

with boundary conditions

$$y(0)=0, \qquad y(1)=1.0 \qquad (2)$$

The constant parameter λ is positive $(\lambda > 0)$. The closed form solution of (1) in terms of the elliptic function $Sc(\langle .|.\rangle)$ of the Jacobian type $[4]$ is given as

$$y(x) = 2/\lambda \sinh^{-1}\left[y'(0)\big/ 2*\left(Sc(\langle \lambda x|m\rangle)\right)\left(1-0.25\left(y'(0)\right)^2\right)\right] \qquad (3)$$

where $y'(0)=2(1-\varsigma)^{0.5}$, and $m=1-0.25\left(y'(0)\right)^2$, ς is the solution of the transcendental equation $\sinh(\lambda/2)=(1-\varsigma)^{0.5}Sc(\langle \lambda|\varsigma\rangle)$. Hence the solution profile $y(x)$ has a singularity located at a pole of $Sc(\langle \lambda\varsigma|m\rangle)$ or approximately

$$x_s = (1/\lambda)\ln\left(\frac{8}{y'(0)}\right) \qquad (4)$$

The implication of (4) is that the singularity lies within the solution range of $y'(0) > 8e^{-\lambda}$. This results in a huge numerical challenge which is further compounded by any increase in λ. However because of Troech's equation application in many fields of engineering and physical sciences, many studies have evolved towards its solution for example, the optimal homotopy asymptotic method[5], quasi-linearization method [6], modified decomposition technique[7], and methods based on Legendre functions [8]. Among these approaches, we hasten to mention the method of transformation [9], the invariant imbedding and the shooting method [10], the homotopy perturbation method [11], quasi-linearization Bessel approach[12] as well as the sinc-Galerkin technique [13]. A common feature of some of these apart from their complicated formulations and tedious application is their inability to produce reliable results for $\lambda > 1$ or even for moderate values of λ . Their inability to deal efficiently with the existence of a pole in the solution profile leaves room for the introduction of mathematical artifacts to facilitate solutions..

The work reported herein seeks to address this challenge by introducing a reliable, efficient and very simple technique that accurately solves the nonlinear Troesch model problem for very large values of λ . As previously mentioned, the hyperbolic-type nonlinearity of the Troesch problem is converted into its polynomial analog by a variable transformation with the sole purpose of dealing with nonlinearity effectively. This idea was firstly reported in [14].and has been modified and applied to the solution of the Troesch problem herein by deploying a straightforward shooting- bisection approach to solve the transformed equation.

II. Mathematical Formulation

For a variable transformation

$$u(x) = \tanh\left(\lambda y(x)/4\right) \qquad (5a)$$

substitute into equation (1)

$$y''(x) = \lambda \sinh\left(4\tanh^{-1}\left(u(x)\right)\right) \qquad (5b)$$

$$y(x) = 4/\lambda \tanh^{-1}\left(u(x)\right) \qquad (5c)$$

$$y'(x) = 4/\lambda\left(u'/\left(1-u^2\right)\right) \qquad (5d)$$

and

$$y'' = 4/\lambda\left(u''/\left(1-u^2\right)\right) + 8/\lambda\left(u\left(u'\right)^2/\left(1-u^2\right)^2\right) \qquad (5e)$$

finally the nonlinear Troesch problem is converted to

$$\left(1-u^2\right)u'' + 2u\left(u'\right)^2 - \lambda^2 u\left(1+u^2\right) = 0 \qquad (5f)$$

using the variable transformation, the corresponding boundary conditions are given as

$$u(0) = 0,\ u(1) = \tanh\left(\lambda/4\right) \qquad (5g)$$

Equation 5(f) can generally be represented as

$$u'' = f\left(x,u,u'\right),\ x \in [a,b],\quad u(a) = \alpha,\ u(b) = \beta \qquad (6a)$$

when we use the shooting method, equation (6a) is transformed into

$$u'' = f\left(x,u,u'\right),\ u(a),\ u'(a) = c \qquad (6b)$$

where c is a guessed or unknown number, which motivates the numerical solution of equation (6a). The final objective here is to finally arrive at c (the slope at the left hand side of the problem domain) such that $u(b,c) = \beta$. In summary we have an objective function defined as a minimization problem

$$G:\mathbb{R} \rightarrow \mathbb{R},\quad G(c) = u(b,c) - \beta \qquad (6c)$$

We search for a critical value of c , the slope at the left hand side boundary of the problem domain such that

$$G\left(c^*\right) = 0 \qquad (6d)$$

For this work, we have decided to employ the simplest of all the shooting methods techniques known as the bisection method. Though it is slow to converge, it often guarantees convergence. We require a preprocessing stage at each level of the computation in order to determine two estimates of the first derivative at the 'left end' boundary of the problem domain where the solution algorithm is initiated. Once we have two estimates below and above the target value at the 'right end' boundary, we apply the bisection method continuously until the tolerance level is met.

The so called 'first guess' is very crucial in the successful implementation of many shooting and iterative solution of nonlinear differential equations. The closer it is to the root the more guaranteed is the convergence. This is even more critical in Newton or the secant methods. We addressed this challenge by solving a linear version of the governing equation and subsequently improved on the first guess by repeatedly employing the bisection method algorithm [15]. The linearized Troesch problem is given as:

$$y'' = \lambda^2 y, \quad y(0) = 0, \quad y(1) = 1 \qquad (6c)$$

whose solution is given as:

$$y(x) = \sinh(\lambda x)/\sinh(\lambda) \qquad (6d)$$

And by direct manipulations

$$u'(0) = \lambda/4\left(y'(0)\right) = \lambda^2/4\left(1/\sinh\lambda\right) \qquad (6e)$$

since $u(0) = 0,$ it follows from equation $5(b)$ that

$$y'(0) = 4/\lambda\left(u'(0)\right) \qquad (6g)$$

As previously inferred, the rate of convergence can differ substantially depending on how close we are to the root We define error convergence as:

$$e_{n+1} = Ce_n^{\xi} \qquad (7a)$$

where C is a constant. The exponent ζ measures how fast the error is diminished from one iteration to the next. The larger the exponent ζ , the faster the error approaches zero and the fewer the iterations we need to arrive at a prescribed stopping criterion $|f(x)| < er$. To estimate ζ , we compute

$$\zeta = \left(\ln\left(e_{n+1}/e_n\right)/\ln\left(e_n/e_{n-1}\right)\right) \qquad (7b)$$

For three consecutive numerical values at $n-1, n, and\ n+1$

III. Discussion of Results

We demonstrate the reliability and robustness of the method presented herein by solving the highly nonlinear Troesch problem for large values of the sensitivity parameter λ . All numerical experiments have been executed using MATTLAB R2017a software.

Figs. 1,2 3 and 4 show the solution profiles for different values of λ . The ability of the model to handle relatively large values of lambda is amply demonstrated.

On the other hand figs. 5 and 6 demonstrate the robustness of the bracketing shooting approach adopted herein. Fig.6 shows the profile of the decrease of the infinity norm error profile between the actual boundary condition of the right hand side of the problem domain and those computed by the shooting technique. The iteration is carried on progressively until the difference falls within the permitted error tolerance which is taken as $10e-06$ for this study

On the other hand Fig. 6 shows the convergence characteristics of the computed right hand side boundary condition. Close observation shows an asymptotic approach to the prescribed right hand side boundary condition after the fifth iteration.

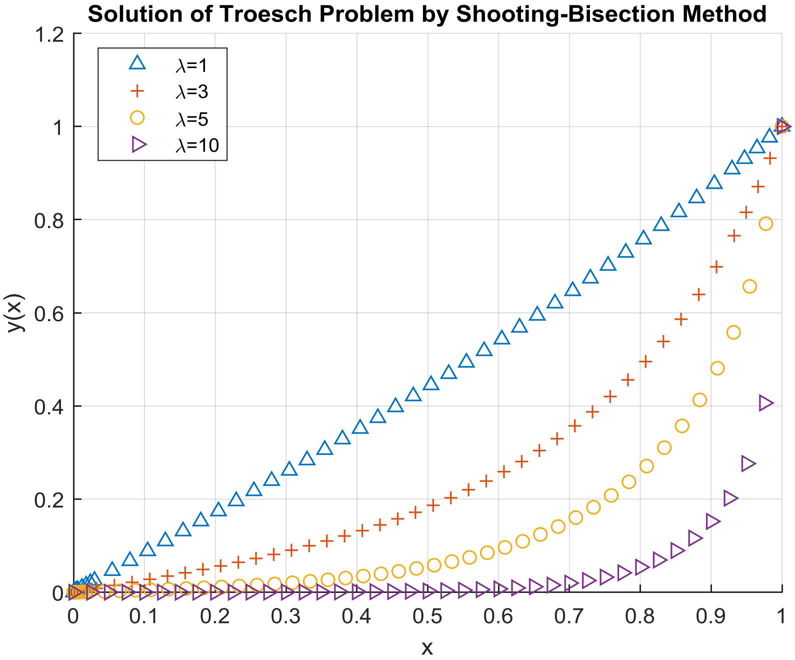

Fig 1 Scalar Profiles for $\lambda = 1, 3, 5$ *and* 10

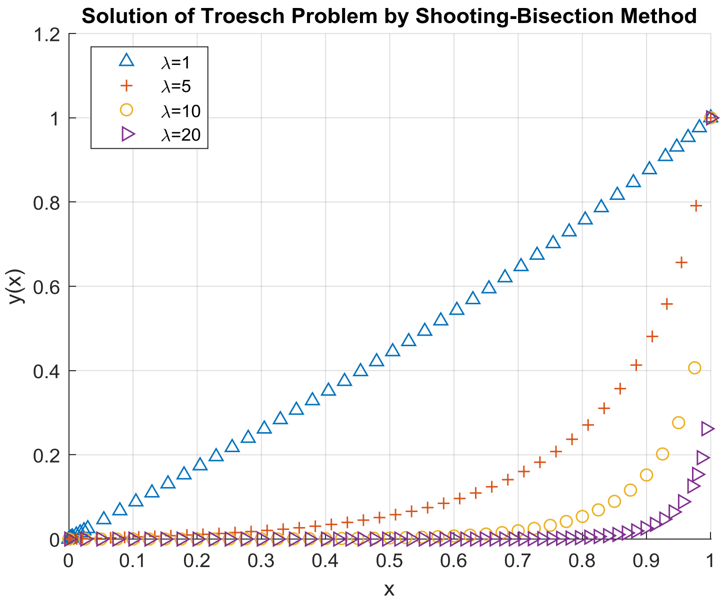

Fig.2 Scalar Profiles for $\lambda = 1, 5, 10$ *and* 20

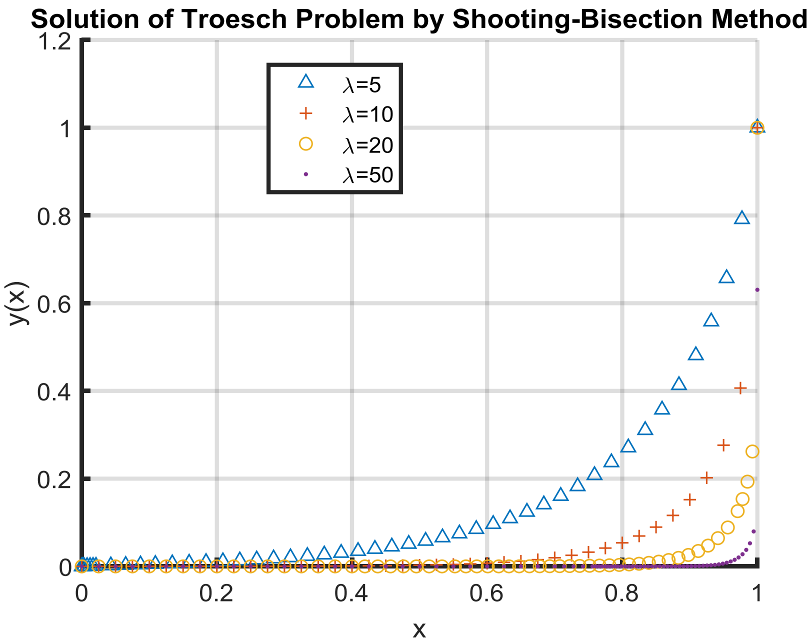

Fig. 3 Scalar Profiles for Scalar Profiles for $\lambda = 5, 10, 20\ and\ 50$

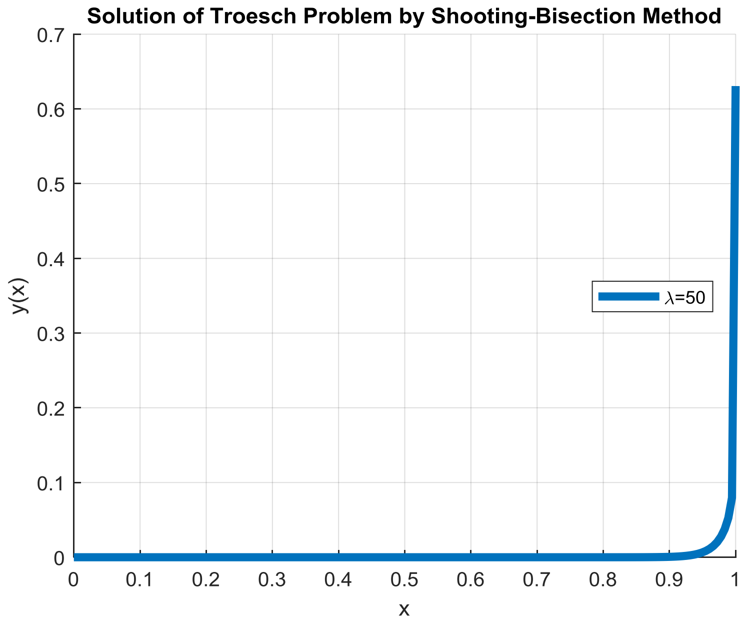

Fig. 4 Scalar Profile for Scalar Profiles for $\lambda = 50$

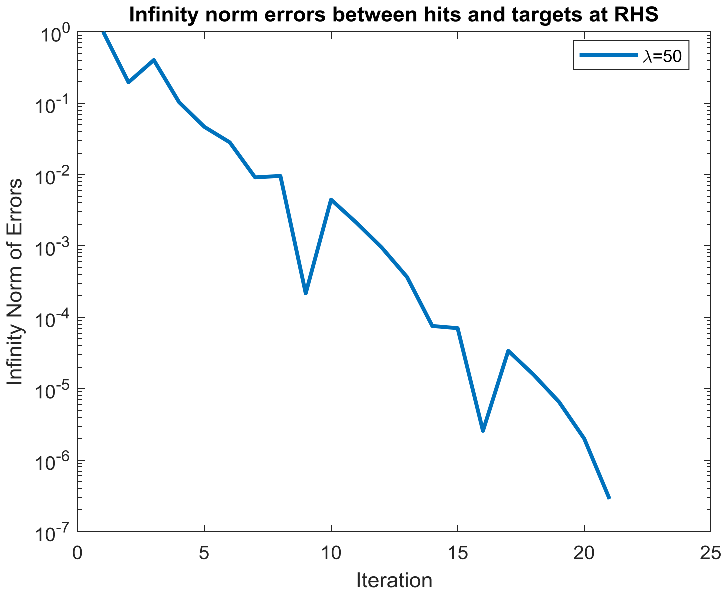

Fig. 5 Infinity norm error profiles between hits and target at the righr hand side boundary

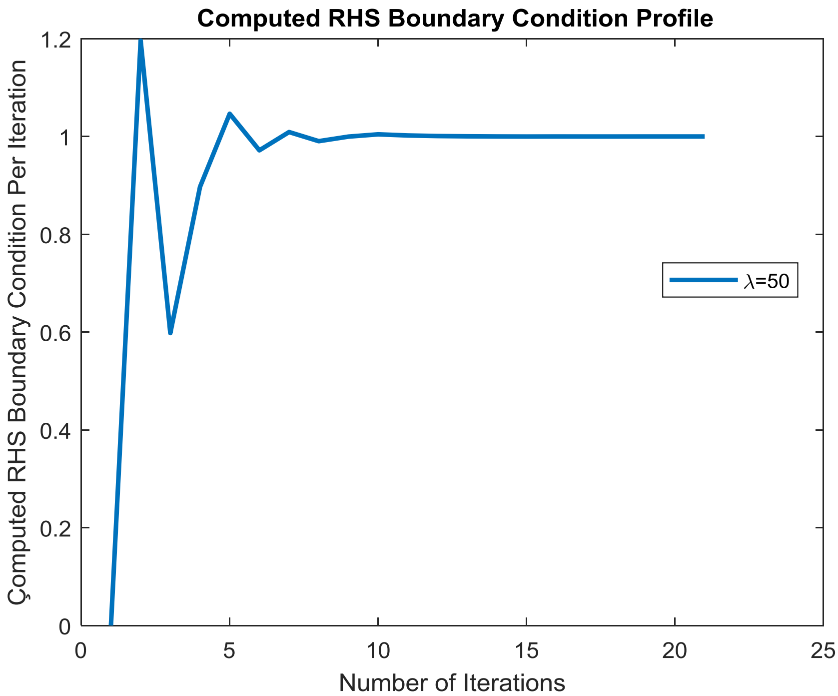

Fig. 6 : Profile of Computed right hand side boundary condition for each iteration

IV. Conclusion

An efficient shooting method technique based on a simple bracketing technique has been applied to solve the nonlinear Troesch problem arising in the modelling of a plasma confinement problem and gas porous electrodes The method was found to yield faithful results for large values of the sensitivity parameter λ in contrast to other more complicated numerical techniques.

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