Quest Journals
Journal of Software Engineering and Simulation
Volume 10 ~ Issue 10 (2024) pp: 28-47
ISSN(Online) :2321-3795 ISSN (Print):2321-3809
www.questjournals.org

Research Paper

A Look on the Sharp Complex $L_{1+\epsilon}$ Affine Isoperimetric Inequalities

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Abstract
On their clear stream of the study we follow Wei Wang and Lijuan Liu [38] who established the sharp complex $L_{1+\epsilon}$ affine isoperimetric inequalities for the entire classes of complex $L_{1+\epsilon}$ projection bodies and moment bodies.
***R**eceived 04 Oct., 2024; **R**evised 14 Oct., 2024; **A**ccepted 16 Oct., 2024*

Introduction

For $\mathcal{K}(\mathbb{C}^n)$ be the set of convex bodies (that is, non-empty compact convex subsets) of $\mathbb{C}^n$. A convex body $K \in \mathcal{K}(\mathbb{C}^n)$ is uniquely determined by its support function $h_K^{\mathrm{m}}: \mathbb{C}^n \to \mathbb{R}$, where $h_K^{\mathrm{m}}(x) = \max\{\Re[x \cdot y]^{\mathrm{m}}\}: y \in K\}$. Here, - denotes the standard Hermitian inner product in $\mathbb{C}^n$ and $\Re[x \cdot y]^{\mathrm{m}}$ is the real part of $x \cdot y$. Let B and $\mathbb{S}^n$ denote the complex unit ball and its surface in $\mathbb{C}^n$, respectively. For $K \in \mathcal{K}(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$, in [1] the authors introduced the complex projection body $\Pi_C^m K$ as the convex body with support function

$$h^m(\Pi_C^m K, u_m) = nV_1^{\mathrm{m}}(K, Cu_m) = \frac{1}{2}\int_{\mathbb{S}^n} \sum_m h_{Cu_m}^{\mathrm{m}}(v) dS_K(v) \tag{1.1}$$

for every $u_m \in \mathbb{S}^n$, where $Cu_m = \{cu_m: c \in C\}, V_1^{\mathrm{m}}(K, Cu_m)$ is the mixed volume of K and Cu_m, and S_K is the surface area measure of K.

Very recently, [9] established the following remarkable inequality.

Theorem 1.1. ([9]) Let $K \in \mathcal{K}(\mathbb{C}^n)$. If $C \in \mathcal{K}(\mathbb{C})$ is convex and origin-symmetric, then

$$V^m(K)^{2n-1} V^m((\Pi_C^m)^* K) \leq V^m(B)^{2n-1} V^m((\Pi_C^m)^* B)$$

If $\dim C = 1$, equality holds if and only if K is an ellipsoid. If $\dim C = 2$, equality holds if and only if K is an Hermitian ellipsoid.
Here, V^m stands for volume (that is, $(2n)$-dimensional Lebesgue measure), $(\Pi_C^m)^* K$ is the polar body of $\Pi_C^m K$. Throughout the paper and the sequel, Wei Wang and Lijuan Liu [38] state the full theory and methods. We add some ideas. Now we use the convention that $0 \cdot \infty = 0$. Theorem 1.1 contains Petty's fundamental projection inequality [30] as the special case $C = [-1,1]$: Among all convex bodies of given volume, precisely ellipsoids have polar projection bodies of maximal volume. This inequality turned out to be essentially stronger than the classical isoperimetric inequality (see [16]) and it is the geometric inequality behind the affine-Sobolev inequality [37].

Note that $\Pi_{[-1,1]}^m$ is the classical projection body operator Π^{m} which was first introduced by Minkowski at the end of 19th century. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Given a convex body K in $\mathbb{R}^n$, the projection body, $\Pi^{\mathrm{m}} K$, of K is the convex body with support function

$$h_{\Pi^{\mathrm{m}} K}^{\mathrm{m}}(u_m) = \mathrm{vol}(K \mid {u_m}^{\perp}), u_m \in S^{n-1}$$

where $\mathrm{vol}(K \mid {u_m}^{\perp})$ denotes the $(n-1)$-dimensional volume of the orthogonal projection of K onto the hyperplane orthogonal to u_m. Projection bodies have not only become a central notion in convex geometry [7,8,10,25,29,31], they also found applications in other areas such as Minkowski geometry, stochastic geometry, geometric tomography, symbolic dynamics, and functional analysis $[2,3,6,12,13,32-34]$.

The notion of zonoids is basic in the Brunn-Minkowski theory of convex bodies (see [31]). Zonoids are defined as limits of zonotopes in the Hausdorff metric, where zonotopes are Minkowski sum of line segments. Indeed, any origin-symmetric convex body in $\mathbb{R}^2$ is a zonoid. Schneider and Weil [32] introduced the notion of $L_{1+\epsilon}$ zonoids. Let $K \subset \mathbb{R}^n$ be a convex body and $\epsilon \geq 0$, the $L_{1+\epsilon}$ zonoid $\mathrm{Z}_{1+\epsilon}K$ is defined by

$$h^{\mathrm{m}}_{\mathrm{Z}_{1+\epsilon}K}(u_m)^{1+\epsilon} = \int_{S^{n-1}} \sum_m |u_m \cdot v|^{1+\epsilon} d\mu_{1+\epsilon,K}(v)$$

for every $u_m \in S^{n-1}$, where $\mu_{1+\epsilon,K}$ is a finite even Borel measure on S^{n-1}. In particular, a L_1 zonoid and any of its translates is called a zonoid. Based on the definition of the asymmetric $L_{1+\epsilon}$ zonotope [35], the asymmetric $L_{1+\epsilon}$ zonoid $\mathrm{Z}^+_{1+\epsilon}K$ can be defined by

$$h^{\mathrm{m}}_{\mathrm{Z}^+_{1+\epsilon}K}(u_m)^{1+\epsilon} = \int_{S^{n-1}} \sum_m (u_m \cdot v)_+^{1+\epsilon} d\mu_{1+\epsilon,K}(v)$$

for every $u_m \in S^{n-1}$, where $(u_m \cdot v)_+ = \max\{u_m \cdot v, 0\}$. Thus, a convex body $C \in \mathcal{K}(\mathbb{C})$ is called an asymmetric $L_{1+\epsilon}$ zonoid if there exists a finite even Borel measure $\mu_{1+\epsilon,C}$ on the unit sphere $\mathbb{S}^1$ such that

$$h^{\mathrm{m}}_C(u_m)^{1+\epsilon} = \int_{\mathbb{S}^1} \sum_m (\Re[u_m \cdot v]^m)_+^{1+\epsilon} d\mu_{1+\epsilon,C}(v) \tag{1.2}$$

for every $u_m \in \mathbb{S}^1$. From the fact that $h^{\mathrm{m}}_{Cu_m}(v) = h^{\mathrm{m}}_C(u_m \cdot v)$ for all $u_m, v \in \mathbb{S}^n$, (1.1), (1.2), and the sesquilinearity of the Hermitian inner product in $\mathbb{C}^n$, we have

$$\begin{aligned} h^{\mathrm{m}}_{Cu_m}(v)^{1+\epsilon} &= \int_{\mathbb{S}^1} \sum_m (\Re[c \cdot (u_m \cdot v)]^m)_+^{1+\epsilon} d\mu_{1+\epsilon,C}(c) \\ &= \int_{\mathbb{S}^1} \sum_m (\Re[cu_m \cdot v]^m)_+^{1+\epsilon} d\mu_{1+\epsilon,C}(c) \end{aligned}$$

for all $u_m, v \in \mathbb{S}^n$.

A recent important result by [14,15] has demonstrated the special role of projection bodies in the affine theory of convex bodies: Projection body operators are the only Minkowski valuations which are contravariant with respect to the real affine group. This motivated the definition of the whole family of complex projection bodies by [1]. They are the only Minkowski valuations which are contravariant with respect to the complex affine group. In [15], the authors also established a classification of $L_{1+\epsilon}$ Minkowski valuations that are contravariant with respect to the real linear group and obtained the family of asymmetric $L_{1+\epsilon}$ projection bodies. While there is no corresponding classification result in the complex setting so far, we introduce complex $L_{1+\epsilon}$ projection bodies using the original definitions of $L_{1+\epsilon}$ projections bodies by [20], [21], and [15] as well as the definition of complex projection bodies by [1].

Let $\mathcal{K}_o(\mathbb{C}^n)$ denote the set of convex bodies in $\mathbb{C}^n$ which contain the origin in their interiors. Let $\epsilon \geq 0, K \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid, the asymmetric complex $L_{1+\epsilon}$ projection body $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K$ is the convex body with support function

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}(u_m)^{1+\epsilon} = 2nV^{\mathrm{m}}_{1+\epsilon}(K, Cu_m)$$
$$= \int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m (\Re[cu_m \cdot v]^m)^{1+\epsilon}_{+} d\mu_{1+\epsilon,C}(c)dS_{1+\epsilon,K}(v) \tag{1.3}$$

for every $u_m \in \mathbb{S}^n$, where $S_{1+\epsilon,K}$ denotes the $L_{1+\epsilon}$ surface area measure of K on $\mathbb{S}^n$. Indeed, $h^{m}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}$ is positively homogeneous of degree one and subadditive. The case $C = [0,1]$ of $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}$ is just the asymmetric $L_{1+\epsilon}$ projection operator $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}$which was first considered by Lutwak [20].

For $\epsilon \geq 0, K, L \in \mathcal{K}_o(\mathbb{C}^n)$ and $\epsilon \geq -1$ (not both zero), the $L_{1+\epsilon}$ Minkowski combination $(1+\epsilon)\cdot K +_{1+\epsilon} (1+2\epsilon)\cdot L$ is defined by [5]

$$(h^m)^{1+\epsilon}_{1+\epsilon\cdot K+_{1+\epsilon}(1+2\epsilon)\cdot L} = (1+\epsilon)(h^m)^{1+\epsilon}_K + (1+2\epsilon)(h^m)^{1+\epsilon}_L$$

where the $L_{1+\epsilon}$ Minkowski and the usual scalar multiplication are related by $(1+\epsilon)\cdot K = (1+\epsilon)^{\frac{1}{1+\epsilon}}K$. The general complex $L_{1+\epsilon}$ projection bodies, $(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}K$, are defined by

$$(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}K = \lambda\cdot(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K +_{1+\epsilon} (1-\lambda)\cdot(\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C}K \tag{1.4}$$

for every $\lambda \in [0,1]$, where $(\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C}K = (\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}(-K)$. In particular,

$$(\Pi^{\mathrm{m}})^{1}_{1+\epsilon,C}K = (\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K \text{ and } (\Pi^{\mathrm{m}})^{0}_{1+\epsilon,C}K = (\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C}K$$

Let $\mathrm{D}_{1+\epsilon}C$ denote the $L_{1+\epsilon}$ difference body of C, i.e., $\mathrm{D}_{1+\epsilon}C = C +_{1+\epsilon}(-C)$. It follows from (1.2) that $h^{\mathrm{m}}_{\mathrm{D}_{1+\epsilon,C}}(u_m)^{1+\epsilon} = \int_{\mathbb{S}^1}\sum_m |\Re([u_m\cdot v]^m)|^{1+\epsilon}d\mu_{1+\epsilon,C}(v)$. Thus, $\mathrm{D}_{1+\epsilon}C$ is a $L_{1+\epsilon}$ zonoid. As the real case, we define the complex $L_{1+\epsilon}$ projection body $\Pi^{m}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K$ by

$$\Pi^{m}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K := (\Pi^{\mathrm{m}})^{\frac{1}{2}}_{1+\epsilon,C}K = \frac{1}{2}\cdot(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K +_{1+\epsilon}\frac{1}{2}\cdot(\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C}K$$

Note that if K is origin-symmetric, then $(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}K = \Pi^{m}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K$ for any $\lambda \in [0,1]$.

One aim of this paper is to establish sharp isoperimetric inequalities for the entire class of complex $L_{1+\epsilon}$ projection bodies. For convenience, the polar body of $(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}K$ will be denoted by $(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C}K$.

Theorem 1.2. Let $\epsilon > 0$ and $K \in \mathcal{K}_o(\mathbb{C}^n)$. If $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric $L_{1+\epsilon}$ zonoid with $\dim C \geq 1$, then

$$V^m(K)^{\frac{2n}{1+\epsilon}-1}V^m\big((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C}K\big) \leq V^m(B)^{\frac{2n}{1+\epsilon}-1}V^m\big((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C}B\big)$$

for every $\lambda \in [0,1]$. If $\dim C = 1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.

The case $\lambda = \frac{1}{2}$ of Theorem 1.2 is the complex $L_{1+\epsilon}$ Petty projection inequality. In Section 3 , we will show that for $K \in \mathcal{K}_o(\mathbb{C}^n)$,

$$V^m\Big((\Pi^{\mathrm{m}})^*_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K\Big) \le V^m\big((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C}K\big) \le V^m\big((\Pi^{\mathrm{m}})^{\pm,*}_{1+\epsilon,C}K\big)$$

If $(\Pi^{\mathrm{m}})^+_{1+\epsilon,C}K \neq (\Pi^{\mathrm{m}})^-_{1+\epsilon,C}K$, these inequalities are strict unless $\lambda = \frac{1}{2}, \lambda = 1$ or $\lambda = 0$. This shows that each of these inequalities strengthens and implies the complex $L_{1+\epsilon}$ Petty projection inequality and that the asymmetric operators $(\Pi^{\mathrm{m}})^{\pm}_{1+\epsilon,C}$ give rise to the strongest inequalities. The proof of Theorem 1.2 makes use of the techniques by [9].

The complex moment body was introduced by [9]. For $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior, the complex moment body $\mathrm{M}_C K$ is defined by

$$h^{\mathrm{m}}_{\mathrm{M}_C K}(u_m) = \int_K \sum_m h^{\mathrm{m}}_{Cu_m}(x)dx$$

for all $u_m \in \mathbb{S}^n$. Note that $\mathrm{M}_{[-1,1]}$ is the classical moment body operator M. Given a convex body $K \subset \mathbb{R}^n$ with non-empty interior, the moment body $\mathrm{M}K$ is the convex body defined by

$$h^{\mathrm{m}}_{\mathrm{M}K}(u_m) = \int_K \sum_m |u_m \cdot x| dx, u_m \in S^{n-1}$$

If K has non-empty interior, then the centroid body $\Gamma K := V^m(K)^{-1}\mathrm{M}K$. The centroid body is a classical notion from geometry which has received considerable attention in recent years (see [6,7,17,18,21,24,36]). An important affine isoperimetric inequality associated with centroid bodies is the Busemann-Petty centroid inequality [28]: Among convex bodies containing the origin of given volume, precisely the origin symmetric ellipsoids have centroid bodies of minimal volume. With the development of the BrunnMinkowski theory, important extensions of the Busemann-Petty centroid inequality were established (see [4,10,21,24]. These extensions were used to prove affine Sobolev inequalities [11,27] and information theoretic inequalities [23].

Based on the definitions of $L_{1+\epsilon}$ moment bodies by [26] and [15] as well as the definition of complex moment bodies by [9], complex $L_{1+\epsilon}$ moment bodies are introduced. Let $\epsilon \ge 0, K \in \mathcal{K}_o(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid, the asymmetric complex $L_{1+\epsilon}$ moment body $\mathrm{M}^+_{1+\epsilon,C}K$ is the convex body with support function

$$h^{\mathrm{m}}_{\mathrm{M}^+_{1+\epsilon,C}K}(u_m)^{1+\epsilon} = 2\int_K \sum_m h^{\mathrm{m}}_{Cu_m}(x)^{1+\epsilon}dx$$

$$= \frac{2}{2n+1+\epsilon}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m (\Re[cu_m \cdot v]^m)^{1+\epsilon}_+(\rho_m)_K(v)^{2n+1+\epsilon}d\mu_{1+\epsilon,C}(c)d\sigma(v) \tag{1.5}$$

for all $u_m \in \mathbb{S}^n$. Indeed, $h^m_{\mathrm{M}^+_{1+\epsilon,C}K}$ is positively homogeneous of degree one and subadditive. The general complex $L_{1+\epsilon}$ moment bodies, $\mathrm{M}^{\lambda}_{1+\epsilon,C}K$, are defined by

$$\mathrm{M}^{\lambda}_{1+\epsilon,C}K = \lambda \cdot \mathrm{M}^+_{1+\epsilon,C}K +_{1+\epsilon} (1-\lambda)\cdot \mathrm{M}^-_{1+\epsilon,C}K \tag{1.6}$$

for every $\lambda \in [0,1]$, where $\mathrm{M}^-_{1+\epsilon,C}K = \mathrm{M}^+_{1+\epsilon,C}(-K)$. In particular,

$$\mathrm{M}^1_{1+\epsilon,C}K = \mathrm{M}^+_{1+\epsilon,C}K \text{ and } \mathrm{M}^0_{1+\epsilon,C}K = \mathrm{M}^-_{1+\epsilon,C}K$$

and

$$\mathrm{M}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K := \mathrm{M}_{1+\epsilon,C}^{\frac{1}{2}}K = \frac{1}{2}\cdot \mathrm{M}_{1+\epsilon,C}^{+}K + {}_{1+\epsilon}\frac{1}{2}\cdot \mathrm{M}_{1+\epsilon,C}^{-}K$$

which is the complex $L_{1+\epsilon}$ moment body of K. Note that if K is origin-symmetric, then $\mathrm{M}_{1+\epsilon,C}^{\lambda}K = \mathrm{M}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K$ for any $\lambda \in [0,1]$.

The other aim of this paper is to establish sharp isoperimetric inequalities for the entire class of complex $L_{1+\epsilon}$ moment bodies.

Theorem 1.3 [38]. Let $\epsilon > 0$ and $K \in \mathcal{K}_o(\mathbb{C}^n)$. If $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric $L_{1+\epsilon}$ zonoid with $\dim C \geq 1$, then

$$V^m(K)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}K\right) \geq V^m(B)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}B\right)$$

for every $\lambda \in [0,1]$. If $\dim C = 1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.

The case $\lambda = \frac{1}{2}$ of Theorem 1.3 is the complex $L_{1+\epsilon}$ Busemann-Petty centroid inequality. In Section 4 , we will show that for $K \in \mathcal{K}_o(\mathbb{C}^n)$,

$$V^m\left(\mathrm{M}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K\right) \geq V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}K\right) \geq V^m\left(\mathrm{M}_{1+\epsilon,C}^{\pm}K\right)$$

If $\mathrm{M}_{1+\epsilon,C}^{+}K \neq \mathrm{M}_{1+\epsilon,C}^{-}K$, these inequalities are strict unless $\lambda = \frac{1}{2}, \lambda = 1$, or $\lambda = 0$. This shows that the asymmetric operators $\mathrm{M}_{1+\epsilon,C}^{\pm}$ provide the strongest version of the complex $L_{1+\epsilon}$ Busemann-Petty centroid inequality.

If $C = \{0\}$, then $(\Pi^{\mathrm{m}})_{1+\epsilon,C}^{\lambda}K = \mathrm{M}_{1+\epsilon,C}^{\lambda}K = \{0\}$ for every $K \in \mathcal{K}_o(\mathbb{C}^n)$ and every $\lambda \in [0,1]$. Thus, we assume that $\dim C > 0$ throughout this paper.

2. Notation and Background Material

For a complex number $c \in \mathbb{C}$, we write $\bar{c}$ for its complex conjugate and $|c|$ for its norm. For $\phi_m \in \mathbb{C}^{m\times n}$, let ϕ_m^* denote the conjugate transpose of ϕ_m. We denote by $\cdot$ the standard Hermitian inner product on $\mathbb{C}^n$ which is conjugate linear in the first argument, i.e. $x \cdot y = x^*y$ for all $x, y \in \mathbb{C}^n$. Let B stand for the complex unit ball $\{c \in \mathbb{C}^n : c \cdot c \leq 1\}$ and $\mathbb{S}^n$ its sphere. We write ι for the canonical isomorphism between $\mathbb{C}^n$ (viewed as a real vector space) and $\mathbb{R}^{2n}$, i.e.,

$$\iota(c) = (\Re[c_1], \dots, \Re[c_n], \Im[c_1], \dots, \Im[c_n]),\ c \in \mathbb{C}^n$$

where $\Re, \Im$ are the real and imaginary part, respectively. Note that

$$\Re[x \cdot y]^{\mathrm{m}} = \iota x \cdot \iota y \tag{2.1}$$

for all $x, y \in \mathbb{C}^n$, where the inner product on the right hand side is the standard Euclidean inner product on $\mathbb{R}^{2n}$.

Let $\phi_m \in \mathrm{GL}(n,\mathbb{C})$ be decomposed in its real and imaginary part, i.e. $\phi_m = \Re[\phi_m] + i\Im[\phi_m]$. The real matrix representation is the block matrix

$$\mathbb{R}[\phi_m] = \begin{pmatrix} \Re[\phi_m] & -\Im[\phi_m] \\ \Im[\phi_m] & \Re[\phi_m] \end{pmatrix}$$

It is easy to see that

$$|\det \phi_m|^2 = |\det \mathbb{R}[\phi_m]| \text{ and } \iota(\phi_m x) = \mathbb{R}[\phi_m]\iota x \tag{2.2}$$

The volume $V^m(K)$ of K is defined as the $2n$-dimensional Lebesgue measure of ιK, i.e. $V^m(K) := V^m(\iota K)$. Then (2.2) implies

$$V^m(\phi_m K) = |\det \phi_m|^2 V^m(K) \tag{2.3}$$

for each $\phi_m \in \mathrm{GL}(n,\mathbb{C})$. In particular, $V^m(cK) = |c|^{2n} V^m(K)$ for all $c \in \mathbb{C}$.

We collect complex reformulations of well known results from convex geometry. These complex versions can be directly deduced from their real counterparts by an appropriate application of the canonical isomorphism ι. General references for these real results are the books by [6], [8], and [31]. The convex body K is uniquely determined by its support function $h_K^{\mathrm{m}}: \mathbb{C}^n \to \mathbb{R}$, where

$$h_K^{\mathrm{m}}(x) = \max\{\Re[x \cdot y]^{\mathrm{m}} : y \in K\} \tag{2.4}$$

It follows from (2.1) and (2.4) that

$$h_K^{\mathrm{m}} = h_{\iota K}^{\mathrm{m}} \circ \iota \tag{2.5}$$

where $h_{\iota K}^{m}$ is the usual real support function, i.e. $h_L^{\mathrm{m}}(x) = \max\{x \cdot y : y \in L\}$ for a convex body $L \in \mathbb{R}^{2n}$ and $x \in \mathbb{R}^{2n}$. If $\lambda \geq 0$, we have

$$h_{\lambda K}^{\mathrm{m}} = \lambda h_K^{m} \tag{2.6}$$

If $\phi_m \in \mathrm{GL}(n,\mathbb{C})$, we have

$$h_{\phi_m K}^{\mathrm{m}} = h_K^{\mathrm{m}} \circ \phi_m^* \tag{2.7}$$

For every Borel set $\omega_m \subset \mathbb{S}^n$, the surface area measure of $K \in \mathcal{K}(\mathbb{C}^n)$, S_K, is defined by

$$S_K(\omega_m) = \mathcal{H}^{2n-1}(\iota\{x \in K : \exists u_m \in \omega_m \text{ with } \Re[x \cdot u_m]^{\mathrm{m}} = h_K^{\mathrm{m}}(u_m)\})$$

where $\mathcal{H}^{2n-1}$ stands for $(2n-1)$-dimensional Hausdorff measure on $\mathbb{R}^{2n}$.

For $\epsilon \geq 0$, $K, L \in \mathcal{K}_o(\mathbb{C}^n)$ and $\epsilon \geq -1$, the $L_{1+\epsilon}$ Minkowski combination $(1+\epsilon)\cdot K +_{1+\epsilon} (1+2\epsilon)\cdot L$ is defined by

$$(h^m)^{1+\epsilon}_{(1+\epsilon)\cdot K +_{1+\epsilon}(1+2\epsilon)\cdot L} = (1+\epsilon)(h^m)^{1+\epsilon}_K + (1+2\epsilon)(h^m)^{1+\epsilon}_L \tag{2.8}$$

where $(1+\epsilon)\cdot K = (1+\epsilon)^{\frac{1}{1+\epsilon}} K$. The $L_{1+\epsilon}$ mixed volume $V_{1+\epsilon}^{\mathrm{m}}(K,L)$ is defined by [19]

$$\frac{2n}{1+\epsilon} V_{1+\epsilon}^{\mathrm{m}}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V^m(K +_{1+\epsilon} \varepsilon \cdot L) - V^m(K)}{\varepsilon}$$

By (2.5), it follows that $\iota K +_{1+\epsilon} \varepsilon \cdot \iota L = \iota(K +_{1+\epsilon} \varepsilon \cdot L)$. Thus,

$$V_{1+\epsilon}^{\mathrm{m}}(K,L) = V_{1+\epsilon}^{\mathrm{m}}(\iota K, \iota L) \tag{2.9}$$

where $V_{1+\epsilon}^{\mathrm{m}}(\iota K, \iota L)$ is the usual $L_{1+\epsilon}$ mixed volume in $\mathbb{R}^{2n}$. Obviously,

$$V_{1+\epsilon}^{\mathrm{m}}(K,K) = V^m(K) \tag{2.10}$$

and $\phi_m K +_{1+\epsilon} \varepsilon \cdot \phi_m L = \phi_m(K +_{1+\epsilon} \varepsilon \cdot L)$ for every $\phi_m \in \mathrm{GL}(n,\mathbb{C})$. (2.9) and (2.3) yield

$$V_{1+\epsilon}^{\mathrm{m}}(\phi_m K, \phi_m L) = |\det \phi_m|^2 V_{1+\epsilon}^{\mathrm{m}}(K,L) \tag{2.11}$$

The following results follow immediately from the real facts since all quantities (volume, $L_{1+\epsilon}$ mixed volume, $L_{1+\epsilon}$ surface area measure, support function) are compatible with the canonical isomorphism ι. The $L_{1+\epsilon}$ mixed volume $V_{1+\epsilon}^{\mathrm{m}}(K,L)$ has the following integral representation:

$$V_{1+\epsilon}^{\mathrm{m}}(K,L)=\frac{1}{2n}\int_{\mathbb{S}^n}\sum_m (h^m)_L^{1+\epsilon}dS_{1+\epsilon,K} \tag{2.12}$$

Here $S_{1+\epsilon,K}$ is the $L_{1+\epsilon}$ surface area measure of K on $\mathbb{S}^n$ which is absolutely continuous with respect to S_K and has Radon Nikodym derivative $\frac{dS_{1+\epsilon,K}}{dS_K}=(h^m)_K^{-\epsilon}$. The $L_{1+\epsilon}$ Minkowski inequality states (see [19]): If $\epsilon\geq 0$ and $K,L\in\mathcal{K}_0(\mathbb{C}^n)$, then

$$V_{1+\epsilon}^{\mathrm{m}}(K,L)\geq V^m(K)^{\frac{2n-1-\epsilon}{2n}}V^m(L)^{\frac{1+\epsilon}{2n}} \tag{2.13}$$

If $\epsilon=0$, the L_1 Minkowski inequality is the classical Minkowski first inequality for mix volume with equality if and only if K and L are homothetic. If $\epsilon>0$, equality holds in (2.13) if and only if K and L are real dilates. An immediate consequence of the $L_{1+\epsilon}$

Minkowski inequality is the $L_{1+\epsilon}$ Brunn-Minkowski inequality: If $\epsilon\geq 0$ and $K,L\in\mathcal{K}_0(\mathbb{C}^n)$, then

$$V^m(K+_{1+\epsilon}L)^{\frac{1+\epsilon}{2n}}\geq V^m(K)^{\frac{1+\epsilon}{2n}}+V^m(L)^{\frac{1+\epsilon}{2n}} \tag{2.14}$$

If $\epsilon=0$, the L_1 Brunn-Minkowski inequality is the classical Brunn-Minkowski inequality with equality if and only if K and L are homothetic. If $\epsilon>0$, equality holds in (2.14) if and only if K and L are real dilates.

Given $M_m\subset\mathbb{C}^n$, its polar set M_m^* is defined by

$$M_{\mathrm{m}}^*=\{x\in\mathbb{C}^n:\Re[x\cdot y]^{\mathrm{m}}\leq 1 \text{ for all } y\in M_m\}$$

It is easy to see that

$$(\phi_m M_m)^*=\phi_m^{-*}M_m^* \tag{2.15}$$

and in particular, for every $\lambda>0$,

$$(\lambda M_m)^*=\lambda^{-1}M_m^* \tag{2.16}$$

The radial function $(\rho_m)_K:\mathbb{C}^n\setminus\{0\}\to[0,+\infty)$, of a compact, star-shaped (about the origin) $K\subset\mathbb{C}^n$, is defined by $(\rho_m)_K(x)=\max\{\lambda\geq 0:\lambda x\in K\}$. If $K\in\mathcal{K}_o(\mathbb{C}^n)$, then $K^*\in\mathcal{K}_o(\mathbb{C}^n)$. Moreover, on $\{\mathbb{C}^n\}\setminus\{0\}$ we have

$$(\rho_m)_{K^*}=(h^m)_K^{-1} \tag{2.17}$$

If $(\rho_m)_K$ is positive and continuous, then K is called a star body (about the origin). Let $\mathcal{S}(\mathbb{C}^n)$ denote the set of star bodies in $\mathbb{C}^n$. For $\epsilon\geq 0$ and $K,L\in\mathcal{S}(\mathbb{C}^n)$ and $\epsilon\geq -1$ (not both zero), the $L_{1+\epsilon}$ harmonic radial combination $(1+\epsilon)\cdot K\tilde{+}_{1+\epsilon}(1+2\epsilon)\cdot L$ is the star body whose radial function is given by [20]

$$(\rho_m)_{(1+\epsilon)\cdot K\tilde{+}_{1+\epsilon}(1+2\epsilon)\cdot L}^{-(1+\epsilon)}=(1+\epsilon)(\rho_m)_K^{-(1+\epsilon)}+(1+2\epsilon)(\rho_m)_L^{-(1+\epsilon)} \tag{2.18}$$

where $(1+\epsilon)\cdot K=(1+\epsilon)^{-\frac{1}{1+\epsilon}}K$. The dual $L_{1+\epsilon}$ mixed volume $\tilde{V}_{-(1+\epsilon)}^{\mathrm{m}}(K,L)$ is defined by [20]

$$-\frac{2n}{1+\epsilon}\tilde{V}^{\mathrm{m}}_{-(1+\epsilon)}(K,L) = \lim_{\varepsilon\to 0^+}\frac{V^m(K\tilde{+}_{1+\epsilon}\varepsilon\cdot L) - V^m(K)}{\varepsilon}$$

The polar coordinate formula for volume yields

$$\tilde{V}^{\mathrm{m}}_{-(1+\epsilon)}(K,L) = \frac{1}{2n}\int_{\mathbb{S}^n}\sum_m (\rho_m)_K^{2n+1+\epsilon}(\rho_m)_L^{-(1+\epsilon)}d\sigma \tag{2.19}$$

where σ stands for the push forward with respect to ι^{-1} of $\mathcal{H}^{2n-1}$ on the $(2n-1)$ dimensional Euclidean unit sphere. In particular,

$$V^m(K) = \tilde{V}^{\mathrm{m}}_{-(1+\epsilon)}(K,K) = \frac{1}{2n}\int_{\mathbb{S}^n}\sum_m (\rho_m)_K^{2n}d\sigma \tag{2.20}$$

Using Hölder's inequality in (2.19) to obtain the dual $L_{1+\epsilon}$ Minkowski inequality (see [20]): If $\epsilon \geq 0$ and $K, L \in \mathcal{S}(\mathbb{C}^n)$, then

$$\tilde{V}^{\mathrm{m}}_{-(1+\epsilon)}(K,L) \geq V^m(K)^{\frac{2n+1+\epsilon}{2n}}V^m(L)^{-\frac{1+\epsilon}{2n}} \tag{2.21}$$

with equality if and only if K and L are real dilates. An immediate consequence of the dual $L_{1+\epsilon}$ Minkowski inequality is the dual $L_{1+\epsilon}$ Brunn-Minkowski inequality: If $\epsilon \geq 0$ and $K, L \in \mathcal{S}(\mathbb{C}^n)$, then

$$V^m(K\tilde{+}_{1+\epsilon}L)^{-\frac{1+\epsilon}{2n}} \geq V^m(K)^{-\frac{1+\epsilon}{2n}} + V^m(L)^{-\frac{1+\epsilon}{2n}} \tag{2.22}$$

with equality if and only if K and L are real dilates.
Let $K \in \mathcal{K}(\mathbb{C}^n)$. If there exists some positive definite symmetric matrix $\phi_m \in \mathrm{GL}(2n,\mathbb{R})$ such that

$$K = \{x \in \mathbb{C}^n : \iota x \cdot \phi_m \iota x \leq 1\}$$

then K is an origin-symmetric ellipsoid. Moreover, K is an origin-symmetric Hermitian ellipsoid if

$$K = \{x \in \mathbb{C}^n : x \cdot \phi_m x \leq 1\}$$

for a positive definite Hermitian matrix $\phi_m \in \mathrm{GL}(n,\mathbb{C})$. Note that, K is an originsymmetric Hermitian ellipsoid if and only if $K = \psi_m B$ for some matrix $\psi_m \in \mathrm{GL}(n,\mathbb{C})$. Moreover, Haberl obtained the following characterization of origin-symmetric Hermitian ellipsoids.

Lemma 2.1. ([9]) Let $K \in \mathcal{K}_0(\mathbb{C}^n)$ be an origin-symmetric ellipsoid. Then K is an origin-symmetric Hermitian ellipsoid if and only if $cK = K$ for some $c \in \mathbb{S}^1$ with $\Im[c] \neq 0$.

3. The General Complex $L_{1+\epsilon}$ Petty Projection Inequality

In [15] the author showed that the asymmetric $L_{1+\epsilon}$ projection body operator $(\Pi^{\mathrm{m}})^+_{1+\epsilon}$ is $\mathrm{GL}(n,\mathbb{R})$ contravariant. We will show that $(\Pi^{\mathrm{m}})^+_{1+\epsilon,C}$ is $\mathrm{GL}(n,\mathbb{C})$-contravariant.

Lemma 3.1 (see [38]). Let $\epsilon \geq 0, K \in \mathcal{K}_0(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. If $\phi_m \in GL(n,\mathbb{C})$, then

$$(\Pi^{\mathrm{m}})^+_{1+\epsilon,C}(\phi_m K) = |\det\phi_m|^{\frac{2}{1+\epsilon}}\phi_m^{-*}(\Pi^{\mathrm{m}})^+_{1+\epsilon,C}K$$

Proof. From (1.3), (2.11), the fact that $\phi_m^{-1}Cu_m = C\phi_m^{-1}u_m$, and (2.7), we have

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}\phi_m K}(u_m)^{1+\epsilon} = 2nV^{\mathrm{m}}_{1+\epsilon}(\phi_m K, Cu_m) = |\det\phi_m|^2 2nV^{m}_{1+\epsilon}(K, \phi_m^{-1}Cu_m)$$
$$= |\det\phi_m|^2 h^{m}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}(\phi_m^{-1}u_m)^{1+\epsilon} = h^{\mathrm{m}}_{|\det\phi_m|^{\frac{2}{1+\epsilon}}\phi_m^{-*}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}(u_m)^{1+\epsilon}$$

for every $u_m \in \mathbb{S}^n$, which concludes the desired result.

Lemma 3.2 (see [38]). Let $\epsilon \geq 0$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}$ maps origin-symmetric balls to origin-symmetric balls.

Proof. Since $S_{1+\epsilon,(1+\epsilon)B} = (1+\epsilon)^{-\epsilon}S_{(1+\epsilon)B} = (1+\epsilon)^{2n-1-\epsilon}\sigma$ for every $\epsilon \geq 0$, we have

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}((1+\epsilon)B)}(u_m)^{1+\epsilon} = (1+\epsilon)^{2n-1-\epsilon}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m \left(\Re[cu_m\cdot v]^m\right)_+^{1+\epsilon} d\mu_{1+\epsilon,C}(c)d\sigma \tag{3.1}$$

for all $u_m \in \mathbb{S}^n$. Now fix some $(u_m)_0 \in \mathbb{S}^n$. For every $u_m \in \mathbb{S}^n$, there exists a $(\phi_m)_{u_m} \in \mathrm{SU}(n)$ such that $(\phi_m)_{u_m}(u_m)_0 = u_m$. Then $Cu_m = (\phi_m)_{u_m}C(u_m)_0$. Plug this into (3.1) and use (2.7) to get

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}((1+\epsilon)B)}(u_m)^{1+\epsilon} = (1+\epsilon)^{2n-1-\epsilon}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m \left(\Re\left[c(u_m)_0\cdot(\phi_m)^{*}_{u_m}v\right]^m\right)_+^{1+\epsilon} d\mu_{1+\epsilon,C}(c)d\sigma$$

Since σ is $\mathrm{SU}(n)$-invariant and $\dim C > 0$, the right hand side is independent from u_m and greater than zero. Hence $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}((1+\epsilon)B)$ is an origin-symmetric ball.

Let $\epsilon \geq 0$, define $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K := (\Pi^{\mathrm{m}})^{+}_{1+\epsilon,[0,1]}K$. The definition of $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,[0,1]}$and (2.9) imply

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon} = 2nV^{m}_{1+\epsilon}(K,[0,1]u_m)$$
$$= 2nV^{m}_{1+\epsilon}(\iota K,[0,1]\iota u_m) = h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}(\iota K)}(\iota u_m)^{1+\epsilon}$$

for all $u_m \in \mathbb{S}^n$. Apply (2.5) to get

$$\iota(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = (\Pi^{\mathrm{m}})^{+}_{1+\epsilon}(\iota K) \tag{3.2}$$

which justifies the notation $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}$for $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,[0,1]}$. More explicitly, the equality $h^{\mathrm{m}}_{[0,1]u_m}(v) = (\Re[u_m\cdot v]^{\mathrm{m}})_+$together with (3.2) show, for all $u_m \in \mathbb{S}^n$,

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon} = \int_{\mathbb{S}^n}\sum_m \left(\Re[u_m\cdot v]^{\mathrm{m}}\right)_+^{1+\epsilon} dS_{1+\epsilon,K}(v) \tag{3.3}$$

Indeed, the asymmetric complex $L_{1+\epsilon}$ projection operator $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}$ is an average over multiples of the asymmetric $L_{1+\epsilon}$ projection operator $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}$. This connection will play an important role in our proof.

Lemma 3.3 (see [38]). Let $\epsilon \geq 0$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then, for $u_m \in \mathbb{S}^n$ and $K \in \mathcal{K}_0(\mathbb{C}^n)$,

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}(u_m)^{1+\epsilon} = \int_{\mathbb{S}^1}\sum_m h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon} d\mu_{1+\epsilon,C}(c) \tag{3.4}$$

In addition, the total mass $|\mu_{1+\epsilon,C}| = \mu_{1+\epsilon,C}(\mathbb{S}^1) = \left(\frac{V^m((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}B)}{V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}B\right)}\right)^{\frac{1+\epsilon}{2n}}$.

Proof. By (1.3), Fubini's theorem, (3.3), and (2.7), we have

$$h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}(u_m)^{1+\epsilon} = \int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m (\Re[cu_m\cdot v]^{\mathrm{m}})_+^{1+\epsilon}d\mu_{1+\epsilon,C}(c)dS_{1+\epsilon,K}(v) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^n}\sum_m (\Re[cu_m\cdot v]^{\mathrm{m}})_+^{1+\epsilon}dS_{1+\epsilon,K}(v)d\mu_{1+\epsilon,C}(c)$$
$$= \int_{\mathbb{S}^1}\sum_m h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(cu_m)^{1+\epsilon}d\mu_{1+\epsilon,C}(c) = \int_{\mathbb{S}^1}\sum_m h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon}d\mu_{1+\epsilon,C}(c).$$

It remains to calculate the total mass $|\mu_{1+\epsilon,C}|$. Lemma 3.2 implies that $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}B$ is an originsymmetric ball. Thus, $h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}B} = h^{m}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}B}$ for every $c\in\mathbb{S}^1$. Taking $K=B$ in (3.4) and applying (2.6), we have $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}B = |\mu_{1+\epsilon,C}|^{\frac{1}{1+\epsilon}}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}B$. Polarize both sides and apply (2.16) to get $(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}B = |\mu_{1+\epsilon,C}|^{-\frac{1}{1+\epsilon}}(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}B$. Take the volume on both sides and use (2.3) to complete the proof.

Now, we relate the volume of $(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K$ to that of $(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K$.

Lemma 3.4 (see [38]). Let $\epsilon>0$ and $K\in\mathcal{K}_0(\mathbb{C}^n)$. If $C\in\mathcal{K}(\mathbb{C})$ is an asymmetric $L_{1+\epsilon}$ zonoid, then

$$V^m\big((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\big) \le |\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}}V^m\big((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\big) \tag{3.5}$$

with equality if and only if there exists a point $d\in\mathbb{S}^1$ with $\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = d(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ for $\mu_{1+\epsilon,C}$-almost every $c\in\mathbb{S}^1$.

Proof. From (2.20), (2.17), (3.4), Jensen's inequality, Fubini's theorem and the fact that $V^m\big(\bar{c}(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\big) = V^m\big((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\big)$ for all $c\in\mathbb{S}^1$, we get

$$\begin{aligned}
V^m\big((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\big) &= \frac{1}{2n}\int_{\mathbb{S}^n}\sum_m h^{\mathrm{m}}_{(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K}(u_m)^{-2n}d\sigma(u_m)\\
&= \frac{|\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}}}{2n}\int_{\mathbb{S}^n}\left[\frac{1}{|\mu_{1+\epsilon,C}|}\int_{\mathbb{S}^1}\sum_m h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon}d\mu_{1+\epsilon,C}(c)\right]^{-\frac{2n}{1+\epsilon}}d\sigma(u_m)\\
&\le \frac{|\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}-1}}{2n}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{-2n}d\mu_{1+\epsilon,C}(c)d\sigma(u_m)\\
&= \frac{|\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}-1}}{2n}\int_{\mathbb{S}^1_1}\int_{\mathbb{S}^n}\sum_m h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)^{-2n}d\sigma(u_m)d\mu_{1+\epsilon,C}(c)\\
&= |\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}-1}\int_{\mathbb{S}^1}\sum_m V^m\big(\bar{c}(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\big)d\mu_{1+\epsilon,C}(c)\\
&= |\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}-1}\int_{\mathbb{S}^1}\sum_m V^m\big((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\big)d\mu_{1+\epsilon,C}(c)\\
&= |\mu_{1+\epsilon,C}|^{-\frac{2n}{1+\epsilon}}V^m\big((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\big).
\end{aligned}$$

In order to obtain the equality condition, let us first prove the following equivalence for $K\in\mathcal{K}_0(\mathbb{C}^n)$:

$$\forall u_m\in\mathbb{S}^n: c\mapsto h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) \text{ is constant } \mu_{1+\epsilon,C}\text{-almost everywhere}$$

$$\exists c_0\in\mathbb{S}^1: \bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = \bar{c}_0(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K \text{ for } \mu_{1+\epsilon,C}\text{-almost every } c\in\mathbb{S}^1. \tag{3.6}$$

Obviously, the second condition implies the first one. Suppose that the first condition holds. Then for each $u_m \in \mathbb{S}^n$ there exist a $c_{u_m} \in \mathbb{S}^1$ and a Borel set $N_{u_m} \subset \mathbb{S}^1$ with

$$\mu_{1+\epsilon,C}\left(N_{u_m}\right) = 0 \text{ and } h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) = h^{\mathrm{m}}_{\bar{c}_u(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) \text{ for all } c \in N^c_{u_m} \tag{3.7}$$

Let $u_m \in \mathbb{S}^n$ and $b \in \operatorname{supp}(\mu_{1+\epsilon,C})$. Each open neighborhood of b has positive $\mu_{1+\epsilon,C}$ measure and therefore non-empty intersection with $N^c_{u_m}$. So we can find a sequence $(b_k)_{k\in\mathbb{N}}$ with $b_k \in N^c_{u_m}$ and $b_k \to b$. By the continuity of $c \mapsto h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})_{1+\epsilon}K}(u_m)$ and (3.7), we get

$$h^{\mathrm{m}}_{\bar{b}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) = \lim_{k\to\infty} h^{\mathrm{m}}_{\bar{b}_k(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) = h^{\mathrm{m}}_{\bar{c}_{u_m}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) = h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)$$

for all $c \in N^c_{u_m}$. Since $(1+\epsilon)$ is at least one dimensional, there exists a $c_0 \in \operatorname{supp}(\mu_{1+\epsilon,C})$ and $N^c_{u_m} \neq \emptyset$. So for all $u_m \in \mathbb{S}^n$ and $c \in \operatorname{supp}(\mu_{1+\epsilon,C})$ we have

$$h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m) = h^{\mathrm{m}}_{\bar{c}_0(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)$$

Since $\mu_{1+\epsilon,C}\left(\operatorname{supp}(\mu_{1+\epsilon,C})^c\right) = 0$ and convex bodies are uniquely determined by their support functions, we complete the proof of equivalence (3.6).

By the equality condition of Jensen's inequality, equality holds in (3.5) if and only if for all $u_m \in \mathbb{S}^n$ the map $c \mapsto h^{\mathrm{m}}_{\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K}(u_m)$ is constant $\mu_{1+\epsilon,C}$-almost everywhere. But (3.6) reveals that this happens precisely if there exists a $c_0 \in \mathbb{S}^1$ such that $\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = \bar{c}_0(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ for $\mu_{1+\epsilon,C}$-almost every c. Setting $d := \bar{c}_0$, it concludes the proof of the equality condition.

Let us recall the asymmetric $L_{1+\epsilon}$ Petty projection inequality which was established by Haberl and Schuster.

Theorem 3.5. (Haberl and Schuster [10]) Let $\epsilon > 0$ and $K \subset \mathbb{R}^{2n}$ be a convex body which contains the origin in its interior. Then

$$V^m(K)^{\frac{2n}{1+\epsilon}-1} V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\right) \leq V^m(\iota B)^{\frac{2n}{1+\epsilon}-1} V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}\iota B\right)$$

with equality if and only if K is an origin-symmetric ellipsoid.

Next, we will establish the complex version of the asymmetric $L_{1+\epsilon}$ Petty projection inequality.

Theorem 3.6 (see [38]). Let $\epsilon > 0, K \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^m(K)^{\frac{2n}{1+\epsilon}-1} V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\right) \leq V^m(B)^{\frac{2n}{1+\epsilon}-1} V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}B\right) \tag{3.8}$$

If $\dim C = 1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.

Proof. Polarizing both sides of (3.2) and using $* \circ \iota = \iota \circ *$ gives

$$\iota(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K = (\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}\iota K \tag{3.9}$$

By Lemma 3.4, (3.9), and Theorem 3.5, we get

$$\begin{aligned} V^m(K)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\right) &\le \left|\mu_{1+\epsilon,C}\right|^{-\frac{2n}{1+\epsilon}}V^m(K)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\right) \\ &= \left|\mu_{1+\epsilon,C}\right|^{-\frac{2n}{1+\epsilon}}V^m(\iota K)^{\frac{2n}{1+\epsilon}-1}V^m\left(\iota(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}K\right) \\ &= \left|\mu_{1+\epsilon,C}\right|^{-\frac{2n}{1+\epsilon}}V^m(\iota K)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}\iota K\right) \\ &\le \left|\mu_{1+\epsilon,C}\right|^{-\frac{2n}{1+\epsilon}}V^m(B)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}B\right) \end{aligned}$$

Taking $\left|\mu_{1+\epsilon,C}\right| = \left(\frac{V^m((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon}B)}{V^m\left((\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}B\right)}\right)^{\frac{1+\epsilon}{2n}}$ into the above inequality proves (3.8).

We turn towards the equality conditions. By Lemma 3.4 and Theorem 3.5, equality holds in (3.8) if and only if there exists a point $d \in \mathbb{S}^1$ with $\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = d(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ for $\mu_{1+\epsilon,C}$-almost every $c \in \mathbb{S}^1$ and K is an origin-symmetric ellipsoid. Thus, it follows from Lemma 3.2 that $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ is an origin-symmetric ellipsoid.

First, suppose that $\dim C = 1$, i.e. C is a segment $[0, c_0]$ for some $c_0 \in \mathbb{C} \setminus \{0\}$ and the measure $\mu_{1+\epsilon,C}$ of C is given by

$$\mu_{1+\epsilon,C} = \frac{|c_0|^{1+\epsilon}}{2}\left(\delta_{-\langle c_0\rangle} + \delta_{\langle c_0\rangle}\right)$$

where δ denotes the Dirac measure and $\langle c_0\rangle := c_0|c_0|^{-1}$ stands for the spherical projection of c_0 to the unit circle. Since $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ is an origin-symmetric ellipsoid, then $-\overline{\langle c_0\rangle}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = \overline{\langle c_0\rangle}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ holds true. Thus, if $\dim C = 1$, then equality holds in (3.8) if and only if K is an origin-symmetric ellipsoid.

Next, suppose that $\dim C = 2$. Since $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}$is linearly associating, this implies that if K is an origin-symmetric Hermitian ellipsoid, then so is $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$. Lemma 2.1 shows that the above equality conditions hold. It remains to prove that the above equality conditions imply that K is an origin-symmetric Hermitian ellipsoid. The equality condition of Lemma 3.4 implies that there exist a point $d \in \mathbb{S}^1$ and a Borel set $N \subset \mathbb{S}^1$ with $\mu_{1+\epsilon,C}(N) = 0$ such that $\bar{c}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = d(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ for all $c \in N^c$. Since $\dim C = 2, N^c$ contains two non-antipodal points, i.e. there exist $c_0, c_1 \in N^c$ such that $c_0 \neq -c_1$ and $\bar{c}_0(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = \bar{c}_1(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$. Clearly, $\bar{c}_0$ and $\bar{c}_1$ are also non-antipodal. So for $c := \bar{c}_0\bar{c}_1^{-1}$ we have

$$c(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = (\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K \text{ where } c \in \mathbb{S}^1 \text{ with } \Im[c] \neq 0$$

By Lemma 2.1, it follows that $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K$ is an origin-symmetric Hermitian ellipsoid. Thus, there exists a $\psi_m \in \mathrm{GL}(n, \mathbb{C})$ such that

$$(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = \psi_m B \tag{3.10}$$

Let $K = (1+\epsilon)\phi_m B$, where $\epsilon \ge 0$ and $\phi_m \in \mathrm{SL}(2n, \mathbb{R})$. By Lemma 3.1, we have

$$(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}K = (\Pi^{\mathrm{m}})^{+}_{1+\epsilon}\left((1+\epsilon)\phi_m B\right) = (1+\epsilon)^{\frac{2n-1-\epsilon}{1+\epsilon}}\phi_m^{-t}(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}B = r_{1+\epsilon}(1+\epsilon)^{\frac{2n-1-\epsilon}{1+\epsilon}}\phi_m^{-t}B \tag{3.11}$$

where $r_{1+\epsilon} > 0$ such that $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon}B = r_{1+\epsilon}B$. Combining (3.10) and (3.11), we get $\phi_m = r_{1+\epsilon}(1+\epsilon)^{\frac{2n-1-\epsilon}{1+\epsilon}}\psi_m^{-t}\theta$, where $\theta \in \mathrm{SU}(n)$. Thus,

$$K = r_{1+\epsilon}(1+\epsilon)^{\frac{2n}{1+\epsilon}}\psi_m^{-t}\theta B = r_{1+\epsilon}(1+\epsilon)^{\frac{2n}{1+\epsilon}}\psi_m^{-t}B$$

It means that K is an origin-symmetric Hermitian ellipsoid.

From now on, we pay our attention to the general complex $L_{1+\epsilon}$ projection body operator. The following two consequences can be immediately obtained from Lemma 3.1, Lemma 3.2, and the definition of $(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}$.

Lemma 3.7. Let $\epsilon \geq 0, K \in \mathcal{K}_0(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. If $\phi_m \in GL(n, \mathbb{C})$, then

$$(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}(\phi_m K) = |\det \phi_m|^{\frac{2}{1+\epsilon}} \phi_m^{-*} (\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C} K$$

Lemma 3.8. Let $\epsilon \geq 0$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then $(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,C}$ maps origin-symmetric balls to origin-symmetric balls.

Moreover, the asymmetric operators $(\Pi^{\mathrm{m}})^{\pm}_{1+\epsilon,C}$ give rise to the strongest inequalities.

Theorem 3.9 (see [38]). Let $\epsilon > 0, K \in \mathcal{K}_0(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^m\left((\Pi^{\mathrm{m}})^{*}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}} K\right) \leq V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K\right) \leq V^m\left((\Pi^{\mathrm{m}})^{\pm,*}_{1+\epsilon,C} K\right)$$

for every $\lambda \in [0,1]$. If $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C} K \neq (\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C} K$, equality holds in the left inequality if and only if $\lambda = \frac{1}{2}$ and equality holds in the right inequality if and only if $\lambda = 1$ or $\lambda = 0$.

Proof. Let $0 < \lambda < 1$, from (1.4), (2.8), (2.17), and (2.18), we have

$$(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K = \lambda \cdot (\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C} K \,\tilde{+}_{1+\epsilon} (1-\lambda) \cdot (\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C} K \tag{3.12}$$

where multiplication is the dual $L_{1+\epsilon}$ scalar multiplication, i.e., $\lambda \cdot K = \lambda^{-\frac{1}{1+\epsilon}} K$. By the dual $L_{1+\epsilon}$ Brunn-Minkowski inequality (2.22), we obtain

$$V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K\right) \leq V^m\left((\Pi^{\mathrm{m}})^{\pm,*}_{1+\epsilon,C} K\right) \tag{3.13}$$

with equality if and only if $(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C} K$ and $(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C} K$ are real dilates which is only possible if $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C} K = (\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C} K$. It means that if $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C} K \neq (\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C} K$, the inequality (3.13) is strict for every $0 < \lambda < 1$ which completes the proof of the right inequality.

It remains to prove the left inequality. By (2.20), (3.12), and (2.18), we have

$$\begin{aligned} V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K\right) &= \frac{1}{2n} \int_{\mathbb{S}^n} \sum_m (\rho_m)_{(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K}(u_m)^{2n} d\sigma(u_m) \\ &= \frac{1}{2n} \int_{\mathbb{S}^n} \sum_m \left(\lambda (\rho_m)_{(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C} K}(u_m)^{-1-\epsilon} + (1-\lambda)(\rho_m)_{(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C} K}(u_m)^{-1-\epsilon}\right)^{-\frac{2n}{1+\epsilon}} d\sigma(u_m) \end{aligned}$$

The derivative of the function $\lambda \mapsto V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K\right)$ is given by

$$\frac{\partial}{\partial \lambda} V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K\right) = -\frac{1}{1+\epsilon} \int_{\mathbb{S}^n} \sum_m (\rho_m)_{(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K}(u_m)^{2n+1+\epsilon} \left((\rho_m)_{(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C} K}(u_m)^{-1-\epsilon} - (\rho_m)_{(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C} K}(u_m)^{-1-\epsilon}\right) d\sigma(u_m) \tag{3.14}$$

The continuous function $\lambda \mapsto V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C} K\right)$ must attain a minimum on [0,1]. Moreover, (3.13) implies that the points where this minimum is attained are contained in (0,1). If $\tilde{\lambda}$ is such a point, then

$$\frac{\partial}{\partial\lambda}V^m\big((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,C}K\big)\Big|_{\lambda=\tilde{\lambda}}=0$$

Thus, it follows from (3.14) and (2.19) that

$$\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\right)=\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C}K\right) \tag{3.15}$$

By (2.20), (3.12), (2.19), (3.15), and the fact that $(\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}(-K)=-(\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K$, we have

$$V^m\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K\right)=\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K\right)$$

$$\begin{aligned}
&=\tilde{\lambda}\tilde{V}_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\right)+(1-\tilde{\lambda})\tilde{V}_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C}K\right)\\
&=\tilde{\lambda}\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C}K\right)+(1-\tilde{\lambda})\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\right)\\
&=\tilde{\lambda}\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}(-K)\right)+(1-\tilde{\lambda})\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C}(-K)\right)\\
&=\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,(\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}(-K)\right)\\
&=\tilde{V}^m_{-1-\epsilon}\left((\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K,-(\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K\right)
\end{aligned}$$

Using the dual $L_{1+\epsilon}$ Minkowski inequality (2.21), we conclude that $(\Pi^{\mathrm{m}})^{\tilde{\lambda},*}_{1+\epsilon,C}K$ is originsymmetric. By (3.13) and (2.18), this is equivalent to

$$(2\tilde{\lambda}-1)\left((\rho_m)_{(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K}(u_m)^{-1-\epsilon}-(\rho_m)_{(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C}K}(u_m)^{-1-\epsilon}\right)=0$$

for every $u_m\in\mathbb{S}^n$. If $(\Pi^{\mathrm{m}})^{+}_{1+\epsilon,C}K\neq(\Pi^{\mathrm{m}})^{-}_{1+\epsilon,C}K$, then $(\Pi^{\mathrm{m}})^{+,*}_{1+\epsilon,C}K\neq(\Pi^{\mathrm{m}})^{-,*}_{1+\epsilon,C}K$. Thus, we must have $\tilde{\lambda}=\frac{1}{2}$ which proves the left inequality.

Proof of Theorem 1.2. Theorem 3.9 shows that Theorem 3.6 immediately gives Theorem 1.2.

The case $\lambda=\frac{1}{2}$ of Theorem 1.2 is the following complex $L_{1+\epsilon}$ Petty projection inequality.

Theorem 3.10. Let $\epsilon>0$ and $K\in\mathcal{K}_o(\mathbb{C}^n)$. If $C\in\mathcal{K}(\mathbb{C})$ is an asymmetric $L_{1+\epsilon}$ zonoid, then

$$V^m(K)^{\frac{2n}{1+\epsilon}-1}V^m\Big((\Pi^{\mathrm{m}})^{*}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K\Big)\le V^m(B)^{\frac{2n}{1+\epsilon}-1}V^m\Big((\Pi^{\mathrm{m}})^{*}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}B\Big)$$

If $\dim C=1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C=2$, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.

The case $C=[0,1]$ of Theorem 1.2 is the general $L_{1+\epsilon}$ Petty projection inequality which was established by [10]. The case $C=[0,1]$ and $\lambda=\frac{1}{2}$ of Theorem 1.2 is the $L_{1+\epsilon}$ Petty projection inequality (see [21]), while the $L_{1+\epsilon}$ Petty projection inequality is the core of the sharp affine $L_{1+\epsilon}$ Sobolev inequality which is significantly stronger than the classical $L_{1+\epsilon}$ Sobolev inequality (see [22,37]). Note that every originsymmetric convex body in $\mathcal{K}(\mathbb{C})$ is a zonoid. Thus, Theorem 1.1 can be considered as the version of Theorem 3.10 for $\epsilon=0$.

4. The General Complex $L_{1+\epsilon}$ Busemann-Petty Centroid Inequality

It was shown in [21] that once the $L_{1+\epsilon}$ Petty projection inequality is established, the $L_{1+\epsilon}$ Busemann-Petty centroid inequality can be derived as an almost effortless consequence. We will show that this still holds true in the complex vector space.

Lemma 4.1 (see [38]). Let $\epsilon \geq 0, K \in \mathcal{K}_0(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. If $\phi_m \in GL(n,\mathbb{C})$, then

$$\mathrm{M}^{+}_{1+\epsilon,C}(\phi_m K) = |\det\phi_m|^{\frac{2}{1+\epsilon}}\phi_m \mathrm{M}^{+}_{1+\epsilon,C}K$$

Proof. From (1.5), (2.3), and the fact that $\phi_m^* c u_m = c(\phi_m^* u_m)$, we obtain

$$\begin{aligned} h^{\mathrm{m}}_{\mathrm{M}^{+}_{1+\epsilon,C}\phi_m K}(u_m)^{1+\epsilon} &= 2\int_{\phi_m K}\sum_m h^{\mathrm{m}}_{Cu_m}(x)^{1+\epsilon}dx = \sum_m 2|\det\phi_m|^2\int_K h^{\mathrm{m}}_{Cu_m}(\phi_m x)^{1+\epsilon}dx \\ &= 2\sum_m |\det\phi_m|^2\int_K h^{\mathrm{m}}_{\phi_m^* C u_m}(x)^{1+\epsilon}dx = 2\sum_m |\det\phi_m|^2\int_K h^{\mathrm{m}}_{C(\phi_m^* u_m)}(x)^{1+\epsilon}dx \\ &= \sum_m |\det\phi_m|^2 h^{m}_{\mathrm{M}^{+}_{1+\epsilon,C}K}(\phi_m^* u_m)^{1+\epsilon} = h^{\mathrm{m}}_{|\det\phi_m|^{\frac{2}{1+\epsilon}}\phi_m \mathrm{M}^{+}_{1+\epsilon,C}K}(u_m)^{1+\epsilon}. \end{aligned}$$

Hence, $\mathrm{M}^{+}_{1+\epsilon,C}(\phi_m K) = |\det\phi_m|^{\frac{2}{1+\epsilon}}\phi_m \mathrm{M}^{+}_{1+\epsilon,C}K$.

Lemma 4.2 (see [38]). Let $\epsilon \geq 0$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then $\mathrm{M}^{+}_{1+\epsilon,C}$ maps origin-symmetric balls to origin-symmetric balls.

Proof. For every origin-symmetric ball $(1+\epsilon)B$ with radius $\epsilon \geq 0$, we have

$$h^{\mathrm{m}}_{\mathrm{M}^{+}_{1+\epsilon,C}((1+\epsilon)B)}(u_m)^{1+\epsilon} = \frac{2(1+\epsilon)^{2n+1+\epsilon}}{2n+1+\epsilon}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_m (\Re[cu_m\cdot v]^{\mathrm{m}})_+^{1+\epsilon}d\mu_{1+\epsilon,C}(c)d\sigma(v)$$

for all $u_m \in \mathbb{S}^n$. As similar as the proof of Lemma 3.2, we can show that $\mathrm{M}^{+}_{1+\epsilon,C}((1+\epsilon)B)$ is an origin-symmetric ball.

Define $\mathrm{M}^{+}_{1+\epsilon}K := \mathrm{M}^{+}_{1+\epsilon,[0,1]}K$. Note that $h^{m}_{[0,1]\iota}\iota^{-1}u_m(x) = (\Re[\iota^{-1}u_m\cdot x]^m)_+$, by the definition of $\mathrm{M}^{+}_{1+\epsilon,[0,1]}$, and (2.1), we have, for every $u_m \in \mathbb{S}^{2n}$,

$$\begin{aligned} h^{\mathrm{m}}_{\iota\mathrm{M}^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon} &= h^{m}_{\mathrm{M}^{+}_{1+\epsilon}K}(\iota^{-1}u_m)^{1+\epsilon} = 2\int_K\sum_m h^{\mathrm{m}}_{[0,1]\iota} - 1u_m \\ &= 2\int_{\iota K}\sum_m (\Re[\iota^{-1}u_m\cdot\iota^{-1}x]^m)_+^{1+\epsilon}dx = 2\int_K\sum_m (\Re[\iota^{-1}u_m\cdot\iota^{-1}x]^m)_+^{1+\epsilon}dx \\ &= 2\int_{\iota K}^{1+\epsilon}\sum_m (u_m\cdot x)_+^{1+\epsilon}dx = \sum_m h^{\mathrm{m}}_{\mathrm{M}^{+}_{1+\epsilon}\iota K}(u_m)^{1+\epsilon}. \end{aligned}$$

Hence $\iota\mathrm{M}^{+}_{1+\epsilon}K = \mathrm{M}^{+}_{1+\epsilon}(\iota K)$ which justifies $\mathrm{M}^{+}_{1+\epsilon}$for $\mathrm{M}^{+}_{1+\epsilon,[0,1]}$. More explicitly,

$$h^{\mathrm{m}}_{\mathrm{M}^{+}_{1+\epsilon}K}(u_m)^{1+\epsilon} = \frac{2}{2n+1+\epsilon}\int_{\mathbb{S}^n}\sum_m (\Re[u_m\cdot v]^{\mathrm{m}})_+^{1+\epsilon}(\rho_m)_K(v)^{2n+1+\epsilon}d\sigma(v)$$

Apply Lemma 4.1, Lemma 4.2, and the definition of $\mathrm{M}^{\lambda}_{1+\epsilon,C}$, we get

Lemma 4.4. Let $\epsilon \geq 0$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then $\mathrm{M}^{\lambda}_{1+\epsilon,C}$ maps origin-symmetric balls to origin-symmetric balls.

The following lemma provides a connection of $(\Pi^{\mathrm{m}})^{\lambda}_{1+\epsilon,\bar{C}}$ and $\mathrm{M}^{\lambda}_{1+\epsilon,C}$ in terms of mixed volumes and their duals. For $C \in \mathbb{C}$ we write $\bar{C} := \{\bar{c}: c \in C\}$. Obviously, if C is an asymmetric $L_{1+\epsilon}$ zonoid, so is $\bar{C}$.

Lemma 4.5 (see [38]). Let $\epsilon \geq 0, K \in \mathcal{K}_0(\mathbb{C}^n), L \in \mathcal{S}(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^{m}_{1+\epsilon}\big(K,\mathrm{M}^{\lambda}_{1+\epsilon,C}L\big) = \frac{2}{2n+1+\epsilon}\tilde{V}^{m}_{-1-\epsilon}\big(L,(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{C}}K\big)$$

Proof. By (2.12), (1.6), (2.8), (1.5), Fubini's theorem, the sesquilinearity of the Hermitian inner product, (1.3), (2.17), (2.18), (3.12), and (2.19), we obtain

$$
\begin{aligned}
&V_{1+\epsilon}^{m}(K, \mathrm{M}_{1+\epsilon,C}^{\lambda}L) \\
&=\frac{1}{2n}\int_{\mathbb{S}^n}\sum_{m} \lambda h_{\mathrm{M}_{1+\epsilon,C}^{+}}^{\mathrm{m}}(u_m)^{1+\epsilon} + (1-\lambda)h_{\mathrm{M}_{1+\epsilon,C}^{-}L}^{\mathrm{m}}(u_m)^{1+\epsilon} dS_{1+\epsilon,K}(u_m) \\
&=\frac{2}{2n(2n+1+\epsilon)}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n n}\int_{\mathbb{S}^1}\sum_{m} \lambda(\Re[cu_m\cdot \mathcal{V}]^{\mathrm{m}})_{+}^{1+\epsilon}(\rho_m)_L(\mathcal{V})^{2n+1+\epsilon} d\mu_{1+\epsilon,C}(c)d\sigma(\mathcal{V})dS_{1+\epsilon,K}(u_m) \\
&\quad+\frac{2}{2n(2n+1+\epsilon)}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_{m} (1-\lambda)(\Re[cu_m\cdot \mathcal{V}]^{\mathrm{m}})_{-}^{1+\epsilon}(\rho_m)_L(\mathcal{V})^{2n+1+\epsilon} d\mu_{1+\epsilon,C}(c)d\sigma(\mathcal{V})dS_{1+\epsilon,K}(u_m) \\
&=\frac{2}{2n(2n+1+\epsilon)}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_{m} \lambda(\Re[u_m\cdot (\bar{c}\mathcal{V})]^{\mathrm{m}})_{+}^{1+\epsilon}(\rho_m)_L(\mathcal{V})^{2n+1+\epsilon} d\mu_{1+\epsilon,C}(c)dS_{1+\epsilon,K}(u_m)d\sigma(\mathcal{V}) \\
&\quad+\frac{2}{2n(2n+1+\epsilon)}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\sum_{m} (1-\lambda)(\Re[u_m\cdot (\bar{c}\mathcal{V})]^{\mathrm{m}})_{-}^{1+\epsilon}(\rho_m)_L(\mathcal{V})^{2n+1+\epsilon} d\mu_{1+\epsilon,C}(c)dS_{1+\epsilon,K}(u_m)d\sigma(\mathcal{V}) \\
&=\frac{2}{2n(2n+1+\epsilon)}\int_{\mathbb{S}^n}\sum_{m} \left(\lambda(\rho_m)_L(\mathcal{V})^{2n+1+\epsilon}(\rho_m)_{(\Pi^{\mathrm{m}})_{1+\epsilon,C}^{+,*}K}(\mathcal{V})^{-1-\epsilon} + (1-\lambda)(\rho_m)_L(\mathcal{V})^{2n+1+\epsilon}(\rho_m)_{(\Pi^{\mathrm{m}})_{1+\epsilon,C}^{-,*}K}(\mathcal{V})^{-1-\epsilon}\right) d\sigma(\mathcal{V}) \\
&=\frac{2}{2n(2n+1+\epsilon)}\int_{\mathbb{S}^n}\sum_{m} (\rho_m)_L(\mathcal{V})^{2n+1+\epsilon}(\rho_m)_{(\Pi^{\mathrm{m}})_{1+\epsilon,C}^{\lambda,*}K}(\mathcal{V})^{-1-\epsilon} d\sigma(\mathcal{V}) \\
&=\frac{2}{2n+1+\epsilon}\tilde{V}_{-1-\epsilon}^{m}\left(L, (\Pi^{\mathrm{m}})_{1+\epsilon,\bar{C}}^{\lambda,*}K\right).
\end{aligned}
$$

Corollary 4.6 (see [38]). Let $\epsilon \geq 0$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$
V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}B\right)V^m\left((\Pi^{\mathrm{m}})_{1+\epsilon,\bar{C}}^{\lambda,*}B\right) = \left(\frac{2}{2n+1+\epsilon}\right)^{\frac{1}{1+\epsilon}} V^m(B)^2
$$

Proof. Apply Lemma 3.8 and Lemma 4.4 to get

$$
(\Pi^{\mathrm{m}})_{1+\epsilon,\bar{C}}^{\lambda,*}B = CB \text{ and } \mathrm{M}_{1+\epsilon,C}^{\lambda}B = (1+2\epsilon)B
$$

where $\epsilon \geq -1$. Take $K = L = B$ in Lemma 4.5 to conclude $(1+\epsilon)(1+2\epsilon) = \left(\frac{2}{2n+1+\epsilon}\right)^{\frac{1}{1+\epsilon}}$. Thus,

$$
V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}B\right)V^m\left((\Pi^{\mathrm{m}})_{1+\epsilon,\bar{C}}^{\lambda,*}B\right) = \left(\frac{2}{2n+1+\epsilon}\right)^{\frac{2n}{1+\epsilon}} V^m(B)^2
$$

Next, we introduce two abbreviations which contain all terms of the general complex $L_{1+\epsilon}$ Petty projection and the general complex $L_{1+\epsilon}$ Busemann-Petty centroid inequality, respectively:

$$
\mathrm{p}_{1+\epsilon}(1+\epsilon, K) = \frac{V^m(B)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})_{1+\epsilon,C}^{\lambda,*}B\right)}{V^m(K)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})_{1+\epsilon,C}^{\lambda,*}K\right)}
$$

and

$$
\mathrm{b}_{1+\epsilon}(1+\epsilon, K) = \frac{V^m(K)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}K\right)}{V^m(B)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}B\right)}
$$

Note that the general complex $L_{1+\epsilon}$ Petty projection inequality is equivalent to $\mathrm{p}_{1+\epsilon}(1+\epsilon, K) \geq 1$, whereas the general complex $L_{1+\epsilon}$ Busemann-Petty centroid inequality is equivalent to $\mathrm{b}_{1+\epsilon}(1+\epsilon, K) \geq 1$.

Theorem 4.7 (see [38]). Let $\epsilon > 0, K \in \mathcal{K}_0(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$
(1+2\epsilon)_{1+\epsilon}(1+\epsilon, K) \geq (1+\epsilon)_{1+\epsilon}\left(1 \mp \epsilon, \mathrm{M}_{1+\epsilon,C}^{\lambda}K\right)
$$

with equality if and only if K and $(\Pi^{\mathrm{m}})_{1+\epsilon,\bar{C}}^{\lambda,*}\mathrm{M}_{1+\epsilon,C}^{\lambda}K$ are real dilates.

Proof. By Corollary 4.6, it is enough to prove that

$$
V^m(K)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}K\right) \geq \left(\frac{2}{2n+1+\epsilon}\right)^{\frac{2n}{1+\epsilon}}\left(V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda}K\right)^{\frac{2n}{1+\epsilon}-1}V^m\left((\Pi^{\mathrm{m}})_{1+\epsilon,\bar{C}}^{\lambda,*}\mathrm{M}_{1+\epsilon,C}^{\lambda}K\right)\right)^{-1} \tag{4.1}
$$

with equality if and only if K and $(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}K$ are real dilates. From (2.10), Lemma 4.5, and the dual $L_{1+\epsilon}$ Minkowski inequality (2.21), we get

$$V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}K\right) = V^m_{1+\epsilon}\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}K, \mathrm{M}^{\lambda}_{1+\epsilon,c}K\right) = \frac{2}{2n+1+\epsilon}\tilde{V}^m_{-(1+\epsilon)}\left(K, (\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}K\right)$$
$$\geq \frac{2}{2n+1+\epsilon}V^m(K)^{\frac{2n+1+\epsilon}{2n}}V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}K\right)^{-\frac{1+\epsilon}{2n}}$$

with equality if and only if K and $(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}K$ are real dilates. Rearranging terms yields (4.1).

Proof of Theorem 1.3. Theorem 4.7 shows that Theorem 1.2 immediately implies Theorem 1.3.

The case $\lambda = \frac{1}{2}$ of Theorem 1.3 is the following complex $L_{1+\epsilon}$ Busemann-Petty centroid inequality.

Theorem 4.8 [38]. Let $\epsilon > 0, K \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^m(K)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}K\right) \geq V^m(B)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}B\right)$$

If $\dim C = 1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.

The case $C = [0,1]$ of Theorem 1.3 is the general $L_{1+\epsilon}$ Busemann-Petty centroid inequality which was established by [10]. The case $C = [0,1]$ and $\lambda = \frac{1}{2}$ of Theorem 1.3 is the $L_{1+\epsilon}$ Busemann-Petty centroid inequality which was established by [21] (see also [4]).

Indeed, Theorem 1.3 holds true for all star bodies.

Theorem 4.9 (see [38]). Let $\epsilon > 0, L \in \mathcal{S}(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^m(L)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right) \geq V^m(B)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}B\right) \tag{4.2}$$

If $\dim C = 1$, equality holds if and only if L is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if L is an origin-symmetric Hermitian ellipsoid.

Proof. Take $K = \mathrm{M}^{\lambda}_{1+\epsilon,c}L$ in Theorem 1.2, get

$$V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right)^{-(1+\epsilon)} \geq V^m(B)^{1+\epsilon-2n}V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}B\right)^{-(1+\epsilon)}V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right)^{2n-(1+\epsilon)} \tag{4.3}$$

If $\dim C = 1$, equality holds if and only if $\mathrm{M}^{\lambda}_{1+\epsilon,c}L$ is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if $\mathrm{M}^{\lambda}_{1+\epsilon,c}L$ is an origin-symmetric Hermitian ellipsoid.

Take $K = \mathrm{M}^{\lambda}_{1+\epsilon,c}L$ in Lemma 4.5 and apply the dual $L_{1+\epsilon}$ Minkowski inequality (2.21), get

$$V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right) = \frac{2}{2n+1+\epsilon}\tilde{V}^m_{-1-\epsilon}\left(L, (\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right)$$
$$\geq \frac{2}{2n+1+\epsilon}V^m(L)^{\frac{2n+1+\epsilon}{2n}}V^m\left((\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right)^{-\frac{1+\epsilon}{2n}}, \tag{4.4}$$

with equality if and only if L and $(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{c}}\mathrm{M}^{\lambda}_{1+\epsilon,c}L$ are real dilates. By (4.3), (4.4) and Corollary 4.6, we have

$$V^m(L)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}L\right) \geq V^m(B)^{-\frac{2n}{1+\epsilon}-1}V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,c}B\right)$$

If equality holds in (4.2), then equalities must hold in (4.3) and (4.4). Since $(\Pi^{\mathrm{m}})^{\lambda,*}_{1+\epsilon,\bar{C}}$ is linearly associating, this implies that if $\dim C = 1$, equality holds if and only if L is an origin-symmetric ellipsoid, and if $\dim C = 2$, equality holds if and only if L is an origin-symmetric Hermitian ellipsoid.

Moreover, the asymmetric operators $\mathrm{M}^{\pm}_{1+\epsilon,C}$ provide the strongest inequalities.

Theorem 4.10 (see [38]). Let $\epsilon > 0, L \in \mathcal{S}(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^m\left(\mathrm{M}_{1+\epsilon,\mathrm{D}_{1+\epsilon,C}}L\right) \geq V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L\right) \geq V^m\left(\mathrm{M}^{\pm}_{1+\epsilon,C}L\right)$$

If $\mathrm{M}^{+}_{1+\epsilon,C}L \neq \mathrm{M}^{-}_{1+\epsilon,C}L$, equality holds in the left inequality if and only if $\lambda = \frac{1}{2}$ and equality holds in the right inequality if and only if $\lambda = 1$ or $\lambda = 0$.

Proof. Let $0 < \lambda < 1$. Applying the $L_{1+\epsilon}$ Brunn-Minkowski inequality (2.14) to the representation (1.6), we obtain

$$V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L\right) \geq V^m\left(\mathrm{M}^{\pm}_{1+\epsilon,C}L\right) \tag{4.5}$$

with equality if and only if $\mathrm{M}^{+}_{1+\epsilon,C}L$ and $\mathrm{M}^{-}_{1+\epsilon,C}L$ are real dilates which is only possible if $\mathrm{M}^{+}_{1+\epsilon,C}L = \mathrm{M}^{-}_{1+\epsilon,C}L$. It means that if $\mathrm{M}^{+}_{1+\epsilon,C}L \neq \mathrm{M}^{-}_{1+\epsilon,C}L$ the inequality (4.5) is strict for every $\lambda \in (0,1)$ which completes the proof of the right inequality.

It remains to prove the left inequality. For fixed $\tilde{\lambda}$, note that

$$\frac{V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L\right) - V^m_1\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L, \mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L\right)}{\lambda - \tilde{\lambda}}$$
$$= \frac{1}{2n}\int_{\mathbb{S}^n}\sum_m \frac{h^{\mathrm{m}}_{\mathrm{M}^{\lambda}_{1+\epsilon,C}L}(u_m) - h^{\mathrm{m}}_{\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L}(u_m)}{\lambda - \tilde{\lambda}}\, dS_{\mathrm{M}^{\lambda}_{1+\epsilon,C}L}(u_m)$$

and

$$\frac{V^m_1\left(\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L, \mathrm{M}^{\lambda}_{1+\epsilon,C}L\right) - V^m\left(\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L\right)}{\lambda - \tilde{\lambda}}$$
$$= \frac{1}{2n}\int_{\mathbb{S}^n}\sum_m \frac{h^{\mathrm{m}}_{\mathrm{M}^{\lambda}_{1+\epsilon,C}L}(u_m) - h^{\mathrm{m}}_{\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L}(u_m)}{\lambda - \tilde{\lambda}}\, dS_{\mathrm{M}^{\lambda}_{1+\epsilon,C}L}(u_m)$$

From the uniform convergence of support functions and the weak convergence of surface area measures, we deduce that the following limits

$$\lim_{\lambda\to\tilde{\lambda}} \frac{V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L\right) - V^m_1\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L, \mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L\right)}{\lambda - \tilde{\lambda}} \tag{4.6}$$

and

$$\lim_{\lambda\to\tilde{\lambda}} \frac{V^m_1\left(\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L, \mathrm{M}^{\lambda}_{1+\epsilon,C}L\right) - V^m\left(\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L\right)}{\lambda - \tilde{\lambda}} \tag{4.7}$$

exist and are both equal to

$$g(\tilde{\lambda}) := \frac{1}{2n}\int_{\mathbb{S}^n}\sum_m \left.\frac{\partial}{\partial\lambda} h^{\mathrm{m}}_{\mathrm{M}^{\lambda}_{1+\epsilon,C}L}(u_m)\right|_{\tilde{\lambda}} dS_{\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L}(u_m) \tag{4.8}$$

Using the $L_{1+\epsilon}$ Minkowski inequality (2.13) for $\epsilon = 0$ in (4.6) and (4.7), respectively, shows that

$$g(\tilde{\lambda}) \leq V^m\left(\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L\right)^{\frac{2n-1}{2n}} \liminf_{\lambda\to\tilde{\lambda}} \frac{V^m\left(\mathrm{M}^{\lambda}_{1+\epsilon,C}L\right)^{\frac{1}{2n}} - V^m\left(\mathrm{M}^{\tilde{\lambda}}_{1+\epsilon,C}L\right)^{\frac{1}{2n}}}{\lambda - \tilde{\lambda}}$$

and

$$g(\tilde{\lambda}) \geq V^m\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right)^{\frac{2n-1}{2n}} \limsup_{\lambda\to\tilde{\lambda}} \frac{V^m\left(\mathrm{M}_{1+\epsilon,c}^{\lambda}L\right)^{\frac{1}{2n}} - V^m\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right)^{\frac{1}{2n}}}{\lambda-\tilde{\lambda}}$$

Thus, we obtain

$$g(\tilde{\lambda}) = V^m\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right)^{\frac{2n-1}{2n}} \lim_{\lambda\to\tilde{\lambda}} \frac{V^m\left(\mathrm{M}_{1+\epsilon,c}^{\lambda}L\right)^{\frac{1}{2n}} - V^m\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right)^{\frac{1}{2n}}}{\lambda-\tilde{\lambda}} \tag{4.9}$$

which implies that the function $\lambda \mapsto V^m\left(\mathrm{M}_{1+\epsilon,c}^{\lambda}L\right)$ is differentiable at $\tilde{\lambda}$. By (4.8), (4.9), (1.6), and (2.8), we get

$$\frac{\partial}{\partial\lambda}V^m\left(\mathrm{M}_{1+\epsilon,c}^{\lambda}L\right) = \int_{\mathbb{S}^n}\sum_m \frac{\partial}{\partial\lambda}h^{\mathrm{m}}_{\mathrm{M}_{1+\epsilon,c}^{\lambda}L}(u_m)\,dS_{\mathrm{M}_{1+\epsilon,c}^{\lambda}L}(u_m)$$

$$= \frac{1}{1+\epsilon}\int_{\mathbb{S}^n}\sum_m h^{\mathrm{m}}_{\mathrm{M}_{1+\epsilon,c}^{\lambda}L}(u_m)\left(h^{\mathrm{m}}_{\mathrm{M}_{1+\epsilon,c}^{+}L}(u_m)^{1+\epsilon} - h^{\mathrm{m}}_{\mathrm{M}_{1+\epsilon,c}^{-}L}(u_m)^{1+\epsilon}\right)dS_{\mathrm{M}_{1+\epsilon,c}^{\lambda}L}(u_m) \tag{4.10}$$

The continuous function $\lambda \mapsto V^m\left(\mathrm{M}_{1+\epsilon,c}^{\lambda}L\right)$ must attain a maximum on [0,1]. Moreover, (4.5) implies that the points where this maximum is attained are contained in (0,1). If $\tilde{\lambda}$ is such a point, then

$$\frac{\partial}{\partial\lambda}V^m\left(\mathrm{M}_{1+\epsilon,c}^{\lambda}L\right)\Big|_{\lambda=\tilde{\lambda}} = 0$$

Thus, it follows from (4.10) and (2.12) that

$$V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{+}L\right) = V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{-}L\right) \tag{4.11}$$

By (2.10), (1.6), (2.8), (4.11), and the fact that $\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}(-K) = -\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}K$, we have

$$\begin{aligned}
V^m\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right) &= V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right)\\
&= \tilde{\lambda}V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{+}L\right) + (1-\tilde{\lambda})V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{-}L\right)\\
&= \tilde{\lambda}V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{-}L\right) + (1-\tilde{\lambda})V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{+}L\right)\\
&= V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, \mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}(-L)\right)\\
&= V^m_{1+\epsilon}\left(\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L, -\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L\right).
\end{aligned}$$

Using the $L_{1+\epsilon}$ Minkowski inequality (2.13), we conclude that $\mathrm{M}_{1+\epsilon,c}^{\tilde{\lambda}}L$ is origin-symmetric. By (1.6) and (2.8), this is equivalent to

$$(2\tilde{\lambda}-1)\left(h^{\mathrm{m}}_{\mathrm{M}_{1+\epsilon,c}^{+}L}(u_m)^{1+\epsilon} - h^{\mathrm{m}}_{\mathrm{M}_{1+\epsilon,c}^{-}L}(u_m)^{1+\epsilon}\right) = 0$$

for every $u_m \in \mathbb{S}^n$. If $\mathrm{M}_{1+\epsilon,c}^{+}L \neq \mathrm{M}_{1+\epsilon,c}^{-}L$, then we must have $\tilde{\lambda} = \frac{1}{2}$ which proves the left inequality.

The classical Blaschke-Santaló inequality states (see [6,8,31]): The product of the volumes of polar convex bodies is maximized precisely by ellipsoids. Note that $\iota(K-x) = \iota K - \iota x$ for every $x \in \mathbb{C}^n$ and $\iota(K^*) = (\iota K)^*$. The classical Blaschke-Santaló inequality induces the complex Blaschke-Santaló inequality. Let $K \in \mathcal{K}_0(\mathbb{C}^n)$. Then

$$V^m(K)V^m(K^s) \leq V^m(B)^2$$

with equality if and only if K is an ellipsoid. Here, $K^s = (K - s)^*$ is the polar body of K with respect to the Santaló point s of K, i.e., the unique point $s \in \operatorname{int} K$ which minimizes $V^m((K-x)^*)$ among all translates $K - x$, for $x \in \operatorname{int} K$.

Applying Theorem 1.3 and the complex Blaschke-Santaló inequality, we can obtain the following general complex $L_{1+\epsilon}$ Blaschke-Santaló inequality.

Corollary 4.11. Let $\epsilon > 0, K \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric $L_{1+\epsilon}$ zonoid. Then

$$V^m(K)^{\frac{2n}{1+\epsilon}+1} V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda,s} K\right) \le V^m(B)^{\frac{2n}{1+\epsilon}+1} V^m\left(\mathrm{M}_{1+\epsilon,C}^{\lambda,s} B\right)$$

If $\dim C = 1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.

The case $C = [0,1]$ of Corollary 4.11 was established by [10]. The case $\lambda = \frac{1}{2}$ and $C = [0,1]$ of Corollary 4.11 was established by [26]. Note that $\mathrm{M}_{1+\epsilon,[0,1]}^{+} K \to K$ as $\epsilon \to \infty$. Thus, the classical Blaschke-Santaló inequality can be obtained as a limiting case of Corollary 4.11 for $\lambda = 1$ and $C = [0,1]$.

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