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Research Paper

Improved Bounds on Polynomial Spectral Radius with Applications to Stability

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Abstract

We let $f(z)$ be a monic complex polynomial of degree n whose roots $z_i, i = 1, ..., n$ satisfy $|z_1| \leq \cdots \leq |z_n|$. We define the spectral radius of $f(z)$ by $R := |z_n|$ and we let $r := |z_1|$. In this self contained article we will present upper and lower bounds on R and r.

The upper bounds coincide with two induced matrix norms of $f(z)$'s companion matrix, say C. I.e., the norm-one bound, say $R_1 := \|\mathrm{C}\|_1$; Cauchy's bound, say R_C, which is a trivial upper bound on R_1; and, the norm-infinity bound also called Montel's bound $R_M := \|\mathrm{C}\|_\infty$. The importance of Cauchy's bound is in its proof technique that implies the sharper bound $R < R_C$. We will extend Cauchy's proof technique to produce new proofs that sharpen the other two bounds R_1 and R_M.

Noting that these bounds are unnaturally restricted by $R_1, R_C, R_M \geq 1$, we will show how to overcome this restriction by considering the polynomial $f(z;\beta) := \beta^n f(z/\beta)$ whose roots are $\beta z_i, i = 1, ..., n$ and its spectral radius is βR. We denote the upper bounds of $f(z;\beta)$ by $\tilde{R}_1(\beta)$ and $\tilde{R}_M(\beta)$, respectively. Hence, $R \leq R_1(\beta) := \tilde{R}_1(\beta)/\beta$ and $R \leq R_M(\beta) := \tilde{R}_M(\beta)/\beta$. We call $R_1(\beta)$ and $R_M(\beta)$ the β-method upper bounds on R which for $\beta > 1$ are at least $1/\beta$, thus removing the restriction that $R_1, R_M \geq 1$. We provide an example for which $R < 1$, $R_1(\beta_1), R_M(\beta_2) < 1$, $\beta_1 \neq \beta_2$, and $\beta_1, \beta_2 > 1$.

Next, we will focus on real monic polynomials and show how to improve the bounds R_1, R_C, and R_M. The idea is to multiply $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ by a series of monic real polynomials, say $g_m(z) = z^m + x_{m-1}z^{m-1} + \cdots + x_1 z + x_0, m = 1, 2, ..., M$. We let $\mathrm{x}_m := [x_0, x_1.., x_{m-1}]^T$, thus $R_u \in \{R_1, R_C, R_M\}$ associated with $h(z) := g_m(z)f(z)$ depends explicitly on x_m, i.e., $R_u \equiv R_u(\mathrm{x}_m)$. The improved bounds will be obtained by minimizing $R_u(\mathrm{x}_m)$ with respect to x_m which turns out to be a set of linear programming (LP) problems. The justification to minimize $R_u(\mathrm{x}_m)$ is because the feasible solution $\mathrm{x}_m = 0$ for which $h(z) = z^m f(z)$ preserves R. We thus arrive at a series of LP problems of increasing complexity that yield a series of non-increasing bounds on $R_u \in \{R_C, R_M\}$, say $R_u \geq R_u^{(1)} \geq R_u^{(2)} \geq \cdots \geq R_u^{(M)}$. For R_1 we could show that $R_1^{(1)} \geq R_1^{(2)} \geq \cdots \geq R_1^{(M)}$ and $R_1 < R_1^{(1)}$ or vice versa. We will present three examples (i) demonstrating the improvements obtained by the LP-method; (ii) an explicit solution of the LP problem for $m = 1$; and, (iii) an example for which the LP-method does not improve R_1 for $m \leq 3$. We will also show how to further improve the LP-method by combining it with the β-method. Notice that $R_u([x_0])$ pertains to the LP-method, where $\mathrm{x}_1 = [x_0]$ and $R_u(\beta)$ without the brackets pertains to the β-method.

A special case of the LP-method is the beautiful Kakeya Theorem, i.e., if $g_1(z) = z - 1$ and $1 \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0$ then $R_M^{(1)} = R_M([-1]) = 1$ is a global minimum of $R_M(\mathrm{x}_1)$. We will also show that if $1 \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$ and $a_0 < 0$ then $R_M([-1]) = 1 + 2|a_0|$. Next, if $g_1(z) = z + 1$ we obtain another Kakeya-type Theorem, where $R_M^{(1)} = R_M([1]) = 1$ is another global minimum of $R_M(\mathrm{x}_1)$. We will show that this case is associated with $\tilde{f}(z) := (-1)^n f(-z)$ whose zeros $\{-z_1, ..., -z_n\}$ preserve the spectral radius R of $f(z)$. We will also discuss stability issues associated with Kakeya's Theorem when it is applicable for N-dimensional (N-D, N≥1) discrete shift-invariant linear systems. For $N = 1$, Jury's cited book on the z-transform states that if $1 > a_{n-1} > \cdots > a_1 > a_0 \geq 0$ then $R < 1$ and consequently $f(z)$ is stable. For $N \geq 2$ we could sharpen Rudin's Theorem and consequently combine it with Kakeya's Theorem when the latter applies. Finally, we present lower bounds, say $\underline{R}$, on R by using Vieta's formulas which imply that if $\underline{R} \geq 1$ then the 1-D linear time-invariant discrete system corresponding to $f(z)$ is not stable. Simulations reveal that if R increases the condition $\underline{R} \geq 1$ is more often satisfied. The study associated with Vieta's formulas led us to a disturbing phenomena that occurred when we computed the coefficients of a high degree stable polynomial $f(z)$ from its given roots. Specifically, we obtained that many of $f(z)$'s trailing coefficients where in the interval [-eps, eps], where eps=$2.22 \cdot 10^{-16}$ is Matlab's constant. In addition, motivated by simulation results we arrived at two conjectures.

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I. Introduction

We let $f(z) := z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0, a_0 \neq 0$ be a monic complex polynomial whose roots $z_1, z_2, ..., z_n$ satisfy $r := |z_1| \leq z_2 \leq \cdots \leq |z_n| =: R$. Hence, the roots of $f(z)$ reside in the annulus $\{z \in \mathbb{C} : r \leq |z| \leq R\}$, where R is called the spectral radius of $f(z)$. To be more explicit we will sometimes write r_f, R_f instead of r, R, respectively. Notice that $r_f = R_F^{-1}$, where

$$F(z) := z^n f(z^{-1})/a_0. \tag{1}$$

In this article we will assume that $f(z)$ is the characteristic polynomial of a discrete time-invariant linear system and call $f(z)$ stable if $f(z) \neq 0, |z| \geq 1$ iff $R < 1$. The purpose of this article is: (i) To review known upper bounds on R, i.e., the induced matrix norm-one bound R_1 and norm-infinity bound R_M also called Montel's bound. (ii) To present new proofs based on Cauchy's proof technique that sharpen them. (iii) To remove the restriction that these bounds should be at least one and show how to apply them to 1-D stability problems. (iv) To improve these bounds for real polynomials by using linear programming. (v) To review Kakeya's Theorem, extend it, and when it is applicable to show how to apply it to test stability of N-D ($N \geq 2$) shift-invariant discrete linear systems by using a sharpened version of Rudin's Theorem. (vi) To obtain a lower bound on R by using Vieta's formulas and apply it to obtain a sufficient condition for instability of 1-D discrete systems.

The organization of this article is as follows. In Section II we study upper bounds on polynomial spectral radius. In Subsection II-A we review the upper bounds on R that coincide with two induced matrix norms of $f(z)$'s companion matrix. I.e., the norm-one bound R_1 and the norm-infinity bound R_M which is also called Montel's bound, see [1, p. 365]. Cauchy's bound, $R_C \geq R_1$ is a trivial bound on R_1. In Section II-B we present Cauchy's proof for $R < R_C$. We then extend Cauchy's proof technique to produce new proofs to (i) if $|a_0| < 1 + \max\{|a_i|, i = 1, ..., n-1\}$ then $R < R_1$, otherwise $R \leq R_1$; and, (ii) if $\sum_{i=0}^{n-1} |a_i| < 1$ then $R < R_M = 1$, otherwise $R \leq R_M$. In Subsection II-C we will show how to remove the unnatural restriction that $R_1, R_M \geq 1$ and provide an example for which $R < 1$ and both β-method bounds corresponding to R_1 and R_M are less than one, though for distinct values of β.

In Section III we will focus on real monic polynomials and will show how to improve the bounds R_1, R_C, and R_M. The idea is to multiply $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ by a series of monic real polynomials, say $g_m(z) = z^m + x_{m-1}z^{m-1} + \cdots + x_1 z + x_0, m = 1, 2, ..., M$. We let $\mathrm{x}_m := [x_0, x_1, ..., x_{m-1})]^T$ then for $m = 1, 2, ..., M$ $R_u \in \{R_1, R_C, R_M\}$ depends explicitly on x_m, i.e., $R_u \equiv R_u(\mathrm{x}_m)$. The improved bounds will be obtained by minimizing $R_u(\mathrm{x}_m)$ with respect to x_m which turns out to be a linear programming (LP) problem. We thus arrive at a series of LP problems of increasing complexity that yield a series of non-increasing bounds on $R_u \in \{R_C, R_M\}$, say $R_u \geq R_u^{(1)} \geq R_u^{(2)} \geq ... \geq R_u^{(M)}$. For R_1 we could show that $R_1^{(1)} \geq R_1^{(2)} \geq \cdots \geq R_1^{(M)}$ and $R_1 < R_1^{(1)}$ or vice versa. In what follows we will call it the LP-method. We will present three examples (i) demonstrating the improvements obtained by the LP-method; (ii) an explicit solution for the LP problem when $m = 1$, similarly to [4]; and, (iii) an example for which the LP-method does not improve R_1 for $m = 3$. Finally, we will point out how to combine the β-method with the LP-method.

In Section IV we present a special case of the LP-method, i.e., the beautiful Kakeya Theorem. If $g_1(z) = z - 1$ and $1 \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0$ then $R_M^{(1)} = R_M([-1]) = 1$ is a global minimum of $R_M(\mathrm{x}_1)$. We will show that if $1 \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$ and $a_0 < 0$ then $R_M([-1]) = 1 + 2|a_0|$. Next, if $g_1(z) = z + 1$ we obtain another Kakeya type Theorem, where $R_M^{(1)} = R_M([1]) = 1$ is another global minimum of $R_M(\mathrm{x}_1)$. In retrospect, it turned out that this case is associated with $\tilde{f}(z) := (-1)^n f(-z)$ whose zeros $\{-z_1, ..., -z_n\}$ preserve the spectral radius R of $f(z)$. In Subsection IV-A we present stability issues associated with Kakeya's Theorem for N-dimensional (N-D, N≥1) discrete linear systems. For $N = 1$ we have: if $1 > a_{n-1} > \cdots > a_1 > a_0 \geq 0$ then $R < 1$ and consequently $f(z)$ is stable, see [2, p. 116]. For $N \geq 2$ we could sharpen Rudin's Theorem thus enabling us to combine it with Kakeya's Theorem when the latter applies. For a proof of Rudin's Theorem see [5], where we generalized Rudin's Theorem that has a single 1-D condition and Strintzis's Theorem that has N 1-D conditions to have any number of up to N 1-D conditions. Notice that Kakeya's Theorem when applicable can be combined only with Rudin's Theorem.

In Section V we will present lower bounds, say $\underline{R}$, on R by using Vieta's formulas from which we will obtain a sufficient condition for instability. I.e., if $\underline{R} \geq 1$ then $f(z)$ is not stable. Simulations reveal that if R increases then the proposed sufficient condition is more often satisfied. The study of Vieta's formulas led us to a disturbing phenomena. I.e., if we choose all the roots of a stable polynomial $f(z)$ of high degree to lie in $[0.1, 0.9]$, then, say k, of $f(z)$'s computed trailing coefficients lie in [-eps,eps], where eps:=$2.22 \cdot 10^{-16}$ is Matlab's constant. Therefore, $f(z) \approx z^k \tilde{f}(z)$ and $f(z)$ acquired k roots that are close to zero which should not have been there in the first place. Furthermore, simulation results associated with $\underline{R}_f$ of $f(z)$ and $\underline{R}_F$ of $F(z) := z^n f(z^{-1})/a_0$ for which $\underline{R}_F = 1/\bar{r}_f$ led us to the formulation of two conjectures.

Finally, in Section VI we give the conclusion.

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II. Bounds on polynomial spectral radius

In Subsection II-A we review the companion matrix norm bounds. Then, in Subsection II-B we present tighter bounds thereof and new proofs based on Cauchy's proof technique. Finally, in Subsection II-C we will show how to remove the unnatural restriction that $R_1, R_M \geq 1$ and give examples for which $R_1 < 1$ and $R_M < 1$.

A. Bounds on polynomial spectral radius via the companion matrix

Here, we will obtain upper bounds on R by using $f(z)$'s companion matrix and either the induced matrix norm-one or the induced matrix norm-infinity. For the sake of completeness we will arrive at $f(z)$'s companion matrix, say $\mathbf{C}$, and then we will show that its characteristic polynomial is $f(z)$. Assuming that $\tilde{z} \in \{z_1, z_2, ..., z_n\}$ we can arrive at the companion matrix as follows. Since,

$$\tilde{z}^n = -a_{n-1}\tilde{z}^{n-1} - \cdots - a_1\tilde{z} - a_0 \tag{2}$$

we can write

$$\mathbf{C}\tilde{\mathbf{v}} = \tilde{z}\tilde{\mathbf{v}}, \tag{3}$$

where

$$\mathbf{C} := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}, \tag{4}$$

$$\tilde{\mathbf{v}} := [1, \tilde{z}, \tilde{z}^2, \cdots, \tilde{z}^{n-1}]^T, \tag{5}$$

and T denotes transposition.

Here, $\tilde{\mathbf{v}}$ denotes the eigenvector of $\mathbf{C}$ corresponding to the eigenvalue $\tilde{z}$. In the general case, when the roots of $f(z)$ are not necessarily distinct, by expanding the first column of $\det(zI - \mathbf{C})$ and applying induction on n we will prove that $f(z) = \det(z\mathbf{I} - \mathbf{C})$. For any matrix norm we have [1, p. 345]

$$R \leq \|\mathbf{C}\|. \tag{6}$$

The induced matrix norm is defined by $\|\mathbf{C}\| := \max_{\mathbf{v}, \|\mathbf{v}\|=1} \|\mathbf{C}\mathbf{v}\|$, therefore, if $\tilde{z}$ is an eigenvalue of $\mathbf{C}$ and $\tilde{\mathbf{v}}$ is the corresponding unit eigenvector then $\|\mathbf{C}\| \geq \|\mathbf{C}\tilde{\mathbf{v}}\| = |\tilde{z}|\|\tilde{\mathbf{v}}\| = |\tilde{z}|$, or $R \leq \|\mathbf{C}\|$. If we apply the induced matrix norm-one we obtain [1, p. 345]

$$\begin{aligned} R \leq \|\mathbf{C}\|_1 &= \max_{j=1,\ldots,n} \sum_{i=1}^{n} |\mathbf{C}(i,j)| \\ &= \max\{|a_0|, 1 + |a_1|, 1 + |a_2|, ..., 1 + |a_{n-1}|\} \\ &\leq \max\{1 + |a_0|, 1 + |a_1|, 1 + |a_2|, ..., 1 + |a_{n-1}|\}. \end{aligned} \tag{7}$$

We let

$$R_1 := \max\{|a_0|, 1 + |a_1|, 1 + |a_2|, ..., 1 + |a_{n-1}|\} \tag{8}$$

denote the norm-one bound and

$$R_C := \max\{1 + |a_0|, 1 + |a_1|, 1 + |a_2|, ..., 1 + |a_{n-1}|\} \tag{9}$$

denote Cauchy's bound.

If we apply the induced matrix norm-infinity we obtain

$$\begin{aligned} R \leq \|\mathbf{C}\|_\infty &= \max_{j=1,\ldots,n} \sum_{i=1}^{n} |\mathbf{C}(i,j)| \\ &= \max\{1, \sum_{i=0}^{n-1} |a_i|\}. \end{aligned} \tag{10}$$

We let

$$R_M := \max\{1, \sum_{i=0}^{n-1} |a_i|\} \tag{11}$$

denote Montel's bound, see [1, p. 365].

Remark 2.1: Notice that

(i) If $a_0 \geq \max\{1+|a_1|, 1+|a_2|, ..., 1+|a_{n-1}|\}$, then $R_1 = |a_0|$ and $R_C = 1 + |a_0|$.

(ii) If $f(z) = z^k \hat{f}(z)$, Montel's bound R_M gives the same result for both $f(z)$ and $\hat{f}(z)$.

(iii) Both $f(z) = z^k \hat{f}(z), k \geq 1$ and $\hat{f}(z)$ have the same spectral radius. However, for $f(z)$ we obtain $R_1 = R_C$, whereas for $\hat{f}(z)$ when $|a_0| \geq \max\{1+|a_1|, 1+|a_2|, ..., 1+|a_{n-1}|\}$ we obtain $R_C = R_1 + 1$.

(iv) Though Montel's bound seems to be weaker than R_1 by applying it to a polynomial whose coefficients are non-negative and non-increasing we obtain the tight Kakeya bound, $R_M = 1$.

B. New proofs and sharper bounds based on Cauchy's Theorem

In Subsection II-A we proved that $R \leq R_C$. Here, we will first prove the sharper version of Cauchy's bound, i.e. $R < R_C$, see [3, p. 375] who calls it Cauchy's Theorem. Next, based on Cauchy's proof technique we will present new proofs and sharper bounds for R_1 and R_M.

Theorem 2.2: $R < R_C$, where

$$R_C := \max\{1+|a_0|, 1+|a_1|, 1+|a_2|, ..., 1+|a_{n-1}|\}. \tag{12}$$

Proof [3, p. 375].

From the definition of $f(z)$ we obtain

$$z^n = f(z) - (a_{n-1}z^{n-1} + \cdots + a_1 x + a_0). \tag{13}$$

We let

$$\rho := \max\{|a_i|, i = 0, 1, ..., n-1\}, \tag{14}$$

then

$$\begin{aligned} |z|^n &\leq |f(z)| + \rho(|z|^{n-1} + \cdots + |z| + 1) \\ &= |f(z)| + \rho\frac{|z|^n - 1}{|z| - 1}. \end{aligned} \tag{15}$$

We let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \geq \rho + 1 \equiv R_C$. Then,

$$\frac{\rho}{|\alpha| - 1} \leq 1. \tag{16}$$

Hence,

$$|\alpha|^n \leq |f(\alpha)| + |\alpha|^n - 1, \tag{17}$$

or

$$|f(\alpha)| \geq 1 \Rightarrow f(\alpha) \neq 0, \alpha \geq R_C. \tag{18}$$

Hence, $R < R_C$. ■

Theorem 2.3: We let

$$\rho_1 := \max\{|a_i|, i = 1, 2, ..., m-1\} \tag{19}$$

and R_1 be as in (8). Then, $|a_0| < \rho_1 + 1$ implies $R < R_1$, otherwise $R \leq R_1$.

Proof.

Using (13) we obtain

$$\begin{aligned} |z|^n &\leq |f(z)| + \rho_1(|z|^{n-1} + \cdots + |z|) + |a_0| \\ &= |f(z)| + \rho_1|z|\frac{|z|^{n-1} - 1}{|z| - 1} + |a_0|. \end{aligned} \tag{20}$$

We let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \geq R_1 = \max\{|a_0|, \rho_1 + 1\}$. Then,

$$\frac{\rho_1}{|\alpha| - 1} \leq 1. \tag{21}$$

Hence,

$$|\alpha|^n \leq |f(\alpha)| + |\alpha|^n - |\alpha| + |a_0|, \tag{22}$$

or

$$|f(\alpha)| \geq |\alpha| - |a_0|. \tag{23}$$

We have
(i) If $|\alpha| \geq R_1 = \rho_1 + 1 > |a_0|$ then $|f(\alpha)| \geq |\alpha| - |a_0| > 0$, hence $R < R_1$.
(ii) Otherwise, if $|\alpha| > R_1 = |a_0| \geq \rho_1 + 1$ then $|f(\alpha)| \geq |\alpha| - |a_0| > 0$, hence $R \leq R_1$.
■

Finally, we could also sharpen Montel's bound, we have
Theorem 2.4: We let

$$R_M := \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}. \tag{24}$$

If $\sum_{i=0}^{m-1} |a_i| < 1$ then $R < R_M = 1$, consequently $f(z)$ is stable.
If $|\alpha| \geq R_M = \sum_{i=0}^{m-1} |a_i| \geq 1$ then $R \leq R_M$.
Proof.
Using (13) we obtain

$$|z|^n \leq |f(z)| + |a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|. \tag{25}$$

We let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \geq R_M$. Then, $|\alpha| \geq 1$ and $|\alpha|^i, i = 0, 1, ..., n-1$ is non-decreasing.
Hence,

$$|\alpha|^n \leq |f(\alpha)| + |\alpha|^{n-1} \sum_{i=0}^{n-1} |a_i|, \tag{26}$$

or

$$|f(\alpha)| \geq |\alpha|^{n-1} \left(|\alpha| - \sum_{i=0}^{n-1} |a_i| \right). \tag{27}$$

If $\sum_{i=0}^{n-1} |a_i| < 1$ then for $|\alpha| \geq R_M = 1 > \sum_{i=0}^{n-1} |a_i|$ we obtain $|f(\alpha)| > 0$. Hence, $f(\alpha) \neq 0, |\alpha| \geq 1$, or $R < 1$, consequently $f(z)$ is stable.

If $\sum_{i=0}^{n-1} |a_i| \geq 1$ then for $|\alpha| > R_M = \sum_{i=0}^{n-1} |a_i|$ we obtain $|f(\alpha)| > 0$. Hence, $f(\alpha) \neq 0, |\alpha| > R_M$, or $R \leq R_M$. ■

C. Removing the restriction that induced matrix norm bounds are at least one

Using (8) and (11) it can be readily seen that $R_1 \geq 1$ and $R_M \geq 1$ which is an unnatural restriction. We will show how to overcome this problem and give examples that improve these bounds. We let

$$\begin{aligned} f(z; \beta) &:= \beta^n f(\beta^{-1} z) \\ &= z^n + \beta^1 a_{n-1} z^{n-1} + \cdots + \beta^{n-1} a_1 z + \beta^n a_0 \\ &= \beta^n \Pi_{i=1}^n (\beta^{-1} z - z_i), \end{aligned} \tag{28}$$

hence the roots of $f(z; \beta)$ are $\beta z_i, i = 1, ..., n$ and its spectral radius is βR_f. Notice that $f(z) \equiv f(z; 1)$. We let $\tilde{R}_1(\beta)$ and $\tilde{R}_M(\beta)$ denote the norm-one and Montel's bounds for $f(z; \beta)$, respectively. Hence:
(i) Using the norm-one bound (8) we obtain

$$\begin{aligned} R &\leq R_1(\beta) := \tilde{R}_1(\beta)/\beta \\ &= \max\left\{|a_0|\beta^{n-1}, \beta^{-1} + |a_1|\beta^{n-2}, \beta^{-1} + |a_2|\beta^{n-3}, ..., \beta^{-1} + |a_{n-1}|\right\}. \end{aligned} \tag{29}$$

(ii) Using Montel's bound (11) we obtain

$$R \leq R_M(\beta) := \tilde{R}_M(\beta)/\beta := \max\{\beta^{-1}, \sum_{i=0}^{n-1} \beta^{n-i-1} |a_i|\}. \tag{30}$$

Example 2.5: We let $f(z) = (z - 0.5)(z - 0.4)(z + 0.3) = z^3 - 0.6z^2 - 0.07z + 0.06$ be a stable polynomial.
Using (8) we obtain that $R_1 = 1.6$. Hence, we can not conclude whether or not $f(z)$ is stable.
However, by minimizing $R_1(\beta)$ we obtain β and $R_1(3.79) = 0.8639$, hence $f(z)$ is stable.

Using (11) we obtain that $R_M = 1$. By Theorem 2.4, since $1 > 0.6 + 0.07 + 0.06$ we obtain that $f(z)$ is stable. Notice that $R_M(3.79) = 1.7271 > R_M = 1$. However, by minimizing $R_M(\beta)$ we obtain that $\beta = 1.27$ and $R_M(1.27) = 0.787 < R_1(3.79) = 0.8639$.

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III. Improved bounds on real polynomial spectral radius via linear programming

In this Section we focus on real polynomials $f(z)$. It is well known that any complex monic polynomial $f(z)$ of degree n can be converted to a real polynomial $d(z)$ of degree $2n$ that preserves the spectral radius of $f(z)$. Hence, we can apply the proposed method to $d(z)$. We have $d(z) := \bar{f}(z)f(z)$, where $\bar{f}(z) := z^n + \bar{a}_{n-1}z^{n-1} + ... + \bar{a}_1 z + \bar{a}_0, a_n := 1$, and $\bar{a}_k$ denotes the complex conjugate of a_k. Since the roots of $\bar{f}(z)$ are the conjugate of the roots of $f(z)$, the quadratic factors of $d(z) = \Pi_{i=1}^n (z - z_i)(z - \bar{z}_i)$ must be real and, therefore, $d(z)$ must also be a real polynomial.

In what follows we will improve the above upper bounds R_1, R_C, and R_M on R_f by using linear programming. The idea is to multiply the real monic polynomial $f(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0, a_n := 1$ by a series of real monic polynomials $g_m(z) = x_m z^m + x_{m-1}z^{m-1} + ... + x_1 z + x_0, x_m := 1, m = 1, ..., M$ whose coefficients will be determined by solving M linear programming (LP) problems. For $u \in \{C, M\}$ we let $R_u^{(0)} := R_u$ and for $m \geq 1$ we let $R_u^{(m)}$ denote the optimal solution associated with $g_m(z), m = 1, ..., M$. We will show that the series of solutions $R_u^{(m)}, m \geq 0$ thus obtained yields a series of non-increasing upper bounds on $f(z)$'s spectral radius. As for R_1 we will show that only the series of solutions $R_1^{(m)}, m \geq 1$ yields a series of non-increasing upper bounds on $f(z)$'s spectral radius. We will present examples demonstrating the improvements obtained by using the proposed LP-method.

Next, we will present the LP based improved upper bounds.

We let

$$\begin{aligned} h(z) &:= f(z)g_m(z) \\ &= h_{n+m}z^{n+m} + h_{n+m-1}z^{n+m-1} + ... + h_1 z + h_0, \end{aligned} \tag{31}$$

where $h_{n+m} := 1$, and we suppress the dependence of $h(z)$ on m. Then,

$$h_k := \sum_{k=i+j} a_i x_j, i \in \{0, 1, ..., n\}, j \in \{0, 1, ..., m\}. \tag{32}$$

Obviously, $R_h \geq R_f$.

Example 3.1: We let $f(z) = a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, a_5 := 1$ and $g_3(z) = x_3 z^3 + x_2 z^2 + x_1 z + x_0, x_3 := 1$. Hence, using (31), $h(z) = g_3(z)f(z)$ is given by

$$\begin{aligned} h(z) =& z^8 + (a_5x_2 + a_4x_3)z^7 + (a_5x_1 + a_4x_2 + a_3)z^6 + (a_5x_0 + a_4x_1 + a_3x_2 + a_2)z^5 + \\ & (a_4x_0 + a_3x_1 + a_2x_2 + a_1)z^4 + (a_3x_0 + a_2x_1 + a_1x_2 + a_0)z^3 + (a_2x_0 + a_1x_1 + a_0x_2)z^2 + \\ & (a_1x_0 + a_0x_1)z^1 + a_0x_0. \end{aligned} \tag{33}$$

Hence, $h(z)$ can be written in matrix form as

$$\mathbf{h} = \mathbf{A}\mathbf{x} + \mathbf{b}, \tag{34}$$

where

$$\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix}; \mathbf{A} = \begin{bmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \\ a_4 & a_3 & a_2 \\ a_5 & a_4 & a_3 \\ 0 & a_5 & a_4 \\ 0 & 0 & a_5 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}; \text{and, } \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{35}$$

Similarly, for $g_2(z) = x_2 z^2 + x_1 z + x_0, x_2 := 1$ we obtain

$$\begin{aligned} h(z) =& z^7 + (a_5x_1 + a_4x_2)z^6 + (a_5x_0 + a_4x_1 + a_3)z^5 + (a_4x_0 + a_3x_1 + a_2)z^4 + \\ & (a_3x_0 + a_2x_1 + a_1)z^3 + (a_2x_0 + a_1x_1 + a_0)z^2 + (a_1x_0 + a_0x_1)z + a_0x_0. \end{aligned} \tag{36}$$

Hence, $h(z)$ can be written in matrix notation as

$$\mathbf{h} = \mathbf{A}\mathbf{x} + \mathbf{b}, \tag{37}$$

where

$$\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix}; \mathbf{A} = \begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \\ a_2 & a_1 \\ a_3 & a_2 \\ a_4 & a_3 \\ a_5 & a_4 \\ 0 & a_5 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}; \text{and, } \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{38}$$

Finally, for $g_1(z) = x_1 z + x_0, x_1 = 1$ we obtain

$$\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix}; \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} x_0 \end{bmatrix}; \text{and, } \mathbf{b} = \begin{bmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{39}$$

By Example 3.1 and using (31) we obtain for any n and m:

$$\mathbf{h} = \mathbf{A}\mathbf{x} + \mathbf{b}. \tag{40}$$

The following three problems can be converted to linear programming problems, see e.g. [8, pp. 134–135]. However, by using CVX [6] these problems will be solved directly without the need to explicitly convert them to LP problems. We let $\tilde{\mathrm{e}} := [0, \underbrace{1, ..., 1}_{n+m-2}]^T$, $|\mathbf{h}| := \mathrm{abs}(\mathbf{h})$, and $\mathbf{h}(i+1) = h_i$.

Using (8) the norm-one bound for $m \geq 1$ is given by

$$R_1^{(m)} := \min_{\mathbf{x}} \||\mathbf{h}| + \tilde{\mathrm{e}}\|_\infty \tag{41}$$
$$\text{subject to}$$
$$\mathbf{h} = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

Using (9) Cauchy's bound for $m \geq 1$ is given by

$$R_C^{(m)} = 1 + \min_{\mathbf{x}} \|\mathbf{h}\|_\infty \tag{42}$$
$$\text{subject to}$$
$$\mathbf{h} = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

Finally, by using (11) Montel's bound for $m \geq 1$ is given by

$$R_M^{(m)} = \min_{\mathbf{x}} \max\{1, \|\mathbf{h}\|_1\} \tag{43}$$
$$\text{subject to}$$
$$\mathbf{h} = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

Notice that we define $R_u^{(0)} := R_u$ only for $u \in \{C, M\}$.

Remark 3.2: We let $E := \tilde{\mathrm{e}}$, then the cvx code is as follows.

For $m \geq 1$, $R_1^{(m)}$ is the optimal value of:

```
cvx_begin
  variable x(m)
  minimize(max(abs(A*x+b)+E))
  subject to
   % none
 cvx_end
```

For $m \geq 1$, $R_C^{(m)}$ is the optimal value of:

```
cvx_begin
  variable x(m)
  minimize(1+max(abs(A*x+b)))
  subject to
  % none
cvx_end
```

For m, $R_M^{(m)}$ is the optimal value of:

```
cvx_begin
  variable x(m)
  minimize(max(1,sum(abs(A*x+b))))
  subject to
  % none
cvx_end
```

Notice that $\mathbf{h}, \mathbf{A}, \mathbf{x}, \mathbf{b}$, and $\tilde{\mathrm{e}}$ all depend on m, where $m \geq 1$.

Theorem 3.3: We have

(i) $R_u^{(1)} \geq \cdots \geq R_u^{(m)} \geq \cdots \geq R_u^{(M)}$, $u \in \{1, C, M\}$, and

(ii) $R_u = R_u([0]) = R_u^{(0)} \geq R_u^{(1)}$, $u \in \{C, M\}$.

Proof.

(i) Assume that $m \geq 1$. We let $\hat{g}_m(z)$ correspond to the optimal solution $\hat{\mathrm{x}}_m := \arg\min_{\mathbf{x}_m} R_u(\mathrm{x}_m)$

and $R_u^{(m)} := R_u(\hat{\mathbf{x}}_m)$, where $u \in \{1, C, M\}$. Since $z\hat{g}_m(z)$ is a feasible solution of the LP problem $\min_{\mathbf{x}_{m+1}} R_u(\mathbf{x}_{m+1})$ we obtain that $R_u^{(m)} \geq R_u^{(m+1)} := R_u(\hat{\mathbf{x}}_{m+1})$.
(ii) For $u \in \{C, M\}$, $R_u([0])$ corresponds to $g_1(z) = z$, hence we obtain
$R_C^{(0)} := R_C = 1 + \max_{i\in\{0,\ldots,m-1\}} |a_i| = R_C([0]) \geq R_C^{(1)}$ and $R_M^{(0)} := R_M = \max\{1, \sum_{i=0}^{m-1} |a_i|\} = R_M([0]) \geq R_M^{(1)}$. For the norm-one bound we have $R_1 = \max\{|a_0|, 1 + \max_{i\in\{1,\ldots,m-1\}} |a_i|\}$ and $R_1([0]) = \max\{0, 1 + \max_{i\in\{0,\ldots,m-1\}} |a_i|\} \equiv R_C$. Since $R_1 \leq R_C \equiv R_1([0]) \geq R_1^{(1)}$ we cannot claim that $R_1 \geq R_1^{(1)}$. See Example 3.4 where $R_1 > R_1^{(1)}$ and Example 3.5 where $R_1 < R_1^{(3)}$. ■
In what follows we will give three examples.

Example 3.4: We let $f(z) = z^5 + z^4 + 0.2z^3 + 0.1z^2 + 2.4z + 1.9$. We have $R_f = 1.2649$.

(i) Using the norm-one bound (8) we obtain

TABLE I
NORM ONE BOUND, R_1.

m	$g_m(z)$	$R_1^{(m)}$
0		$R_1 = 3.4$
1	$z - 1.7$	3.23
2	$z^2 - 1.055z + 0.1715$	3.2061
3	$z^3 - 1.2508z^2 + 0.8387z - 1.1871$	2.2555

(ii) Using Cauchy's bound (9) we obtain

TABLE II
CAUCHY'S BOUND

m	$g_m(z)$	$R_C^{(m)}$
0	z	3.4
1	$z - 1.2$	3.28
2	$z^2 + 0.1715z - 1.055$	3.2062
3	$z^3 - 1.0323z^2 + 0.2123z - 0.8066$	2.5326

(iii) Finally, using Montel's bound (10) we obtain

TABLE III
MONTEL'S BOUND

m	$g_m(z)$	$R_M^{(m)}$
0	z	5.6
1	$z - 0.7917$	4.68333
2	$z^2 - 0.8174z + 0.6174$	4.42261
3	$z^3 - 0.7762z^2 + 0.6355z - 0.5031$	3.26242

Example 3.5: The column of R_1 need not necessarily be non-increasing as in the previous example. However, the $R_1^{(m)}$'s for $m = 1, \ldots, M$ are non-increasing. This can be seen by considering the following example. We let

$$f(z) = z^3 + 0.4z^2 + 0.4z + 1.6, \tag{44}$$

where $R_f = 1.1914$. For $m = 1$, by using (8) we obtain $R_1 = 1.6$. However, the optimal $g_1(z) = z - 1.3$ gives $R_1^{(1)} = 2.08$ that is larger than R_1. Therefore, we have to use the following correction for the norm-one bounds

$$R_1^{(m)} := \min\{R_1^{(m)}, R_1\}, m = 1, \ldots, M. \tag{45}$$

Example 3.6: The solution proposed here for $m = 1$ is similar to that of [4]. For $m = 1$ we have

$$\begin{aligned} h(z) &= h_{n+1}z^{n+1} + \cdots + h_1 z + h_0 \\ &= z^{n+1} + (a_n x_0 + a_{n-1}) + \cdots + (a_2 x_0 + a_1)z^2 + (a_1 x_0 + a_0)z + a_0 x_0. \end{aligned} \tag{46}$$

It is well known that any optimal value of a LP problem can be attained at one of its extreme points. In this problem the extreme points occur when the argument of an absolute value is zero. Both bounds $R_1([x_0])$ and $R_M([x_0])$ have at most $n + 1$ extreme points. Hence, we obtain

$$\begin{aligned} R_1([x_0]) = &\max\{|h_0|, 1 + |h_1|, 1 + |h_2|, \cdots, 1 + |h_{n-1}|\} \\ &\max\{|a_0 x_0|, 1 + |a_1 x_0 + a_0|, 1 + |a_2 x_0 + a_1|, \ldots, 1 + |a_n x_0 + a_{n-1}|\} \end{aligned} \tag{47}$$

Now, since $a_0 \neq 0$, it may happen that $R_1 \neq R_1([0])$. Therefore, we obtain

$$R_1^{(1)} = \min\{R_1([0]), R_1([-a_0/a_1]), \ldots, R_1([-a_{n-1}/a_n])\},$$

where we define $R_1([-a_i/0]) := \infty$. and for all m such that $R_1 < R_1^{(m)}$ we will output R_1.
Similarly, for $R_M^{(1)}$ we obtain

$$R_M([x_0]) = \max\{1, \sum_{i=0}^{n} |h_i|\} \qquad (48)$$
$$\max\{1, |a_0x_0| + |a_1x_0 + a_0| + |a_2x_0 + a_1| + \cdots + |a_nx_0 + a_{n-1}|\}.$$

Hence,

$$R_M^{(1)} = \min\{R_M([0]), R_M([-a_0/a_1]), ..., R_M([-a_{n-1}/a_n])\}, \qquad (49)$$

where we define $R_M([-a_i/0]) := \infty$.
We will conclude this Section with the following two remarks.

Remark 3.7: Here, we will point out how to combine the LP-method with the β-method. Specifically,

$$\text{opt } R_1^{(m)} := \min\left\{R_1, \min_{\beta}\left[\frac{1}{\beta}\min_{\mathbf{x}_m}\tilde{R}_1(\mathbf{x}_m, \beta)\right]\right\} \qquad (50)$$

and

$$\text{opt } R_M^{(m)} := \min_{\beta}\left\{\frac{1}{\beta}\min_{\mathbf{x}_m}\tilde{R}_M(\mathbf{x}; \beta)\right\}, \qquad (51)$$

where both $\tilde{R}_1(\mathbf{x}_m, \beta)$ and $\tilde{R}_M(\mathbf{x}; \beta)$ are associated with $g_m(z)f(z;\beta)$. Hence, at each point β of the line search we have to solve an LP problem. Notice that for $m = 1$ we can exploit the explicit solution of Example 3.6.

Remark 3.8: Here, we will discuss how to improve the spectral radius of matrices. Suppose $\mathbf{M} \in \mathbb{C}^{n\times n}$ and $f(z) = det(z\mathbf{I} - \mathbf{M})$ is its characteristic polynomial . We let $f(z)$'s roots (also called eigenvalues) be $z_1, ..., z_n$, where $|z_1| \leq ... \leq |z_n| =: R_f$. The spectral radius of $\mathbf{M}$ is R_f and we have $R_f \leq \|\mathbf{M}\|$, see [1, p. 345]. The eigenvalues of $\beta\mathbf{M}$ are the βz_i's and its spectral radius is βR_f. We have $\not\beta R_f \leq \|\not\beta\mathbf{M}\| = \not\beta\|\mathbf{M}\|$, where β cancels out, thus rendering the β-method futile. However, if we proceed with $f(z)$ as in the previous Remark by using the β-method, or the LP-method, or both, we can thus improve M's spectral radius.

IV. Kakeya's Theorem and its extensions

Here, we will present Kakeya's Theorem [1, p. 366], an extension thereof, a new Kakeya-type Theorem, and its application to stability of N-D discrete systems.

Theorem 4.1: Suppose $f(z) = z^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ is a given polynomial with real coefficients that are monotone, in the sense $1 \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0$. Then

(i) Kakeya's Theorem: if $a_0 \geq 0$ then $R \leq R_M([-1]) = 1$.

(ii) Extended Kakeya's Theorem: if $a_0 < 0$ then $R \leq R_M([-1]) = 1 + 2|a_0|$.

Proof.
We let

$$h(z) = (z-1)f(z) \qquad (52)$$
$$= z^{n+1} + (a_{n-1} - 1)z^n + (a_{n-2} - a_{n-1})z^{n-1} + \cdots + (a_0 - a_1)z - a_0.$$

Appying Montel's bound (10) to $h(z)$ and using the monotonicity of $f(z)$'s coefficients we obtain

$$R_M([-1]) = \max\{1, |a_{n-1} - 1| + |a_{n-2} - a_{n-1}| + \cdots + |a_0 - a_1| + |-a_0|\} \qquad (53)$$
$$= \max\{1, [-a_{n-1} + 1] + [-a_{n-2} + a_{n-1}] + \cdots + [-a_0 + a_1] + |a_0|\}$$
$$= \max\{1, 1 - a_0 + |a_0|\}.$$

In case (i) since $a_0 \geq 0$ we obtain $R_M([-1]) = \max\{1, 1 - a_0 + a_0\} = 1$.
In case (ii) since $a_0 < 0$ we obtain $R_M([-1]) = \max\{1, 1 - a_0 - a_0\} = 1 + 2|a_0|$. ■

Theorem 4.2: Suppose $f(z) = z^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ is a given polynomial with real coefficients that are monotone, in the sense $1 \geq (-1)^1a_{n-1} \geq (-1)^2a_{n-2} \geq (-1)^3a_{n-3} \geq \cdots \geq (-1)^{n-1}a_1 \geq (-1)^na_0$. Then
(i) For n odd: if $a_0 < 0$ then $R \leq R_M([1]) = 1$; and, if $a_0 > 0$ then $R \leq R_M([1]) = 1 + 2|a_0|$.
(ii) For n even: if $a_0 > 0$ then $R \leq R_M([1]) = 1$; and, if $a_0 < 0$ then $R \leq R_M([1]) = 1 + 2|a_0|$.

Proof.
We let

$$h(z) = (z+1)f(z) \qquad (54)$$
$$= z^{n+1} + (1 + a_{n-1})z^n + (a_{n-2} + a_{n-1})z^{n-1} + \cdots + (a_0 + a_1)z + a_0.$$

For n odd:

$$R_M([1]) = \max\{1, |a_{n-1} + 1| + |a_{n-1} + a_{n-2}| + \cdots + |a_0 + a_1| + |a_0|\} \qquad (55)$$
$$= \max\{1, [a_{n-1} + 1] + [-a_{n-1} - a_{n-2}] + \cdots + [-a_1 - a_2] + [a_0 + a_1] + |a_0|\}$$
$$= \max\{1, 1 + a_0 + |a_0|\}.$$

Hence, if $a_0 < 0$ then $R_M([1]) = 1$; and, if $a_0 > 0$ then $R_M([1]) = 1 + 2|a_0|$.
For n even:

$$\begin{aligned} R_M([1]) &= \max\{1, |a_{n-1}+1| + |a_{n-2}+a_{n-1}| + \cdots + |a_0+a_1| + |a_0|\} \\ &= \max\{1, [a_{n-1}+1] + [-a_{n-2}-a_{n-1}] + \cdots + [a_1+a_2] + [-a_0-a_1] + |a_0|\} \\ &= \max\{1, a_{n-1}+1+a_{n-2}-a_{n-1} + \cdots + a_0+a_1+|a_0|\} \\ &= \max\{1, 1-a_0+|a_0|\}. \end{aligned} \quad (56)$$

Hence, if $a_0 > 0$ then $R_M([1]) = 1$; and, if $a_0 < 0$ then $R_M([1]) = 1 + 2|a_0|$. ■

Remark 4.3: The above theorem to the best of our knowledge is new and its proof is nontrivial. However, the result is trivial and was overlooked, since we can obtain Theorem 4.2 by applying Theorem 4.1 to $\hat{f}(z) := (-1)^n f(-z)$ and noting that the zeros of $\hat{f}(z)$, $-z_i, i = 1, ..., n$ preserve the spectral radius, i.e., $R_{\hat{f}} = R_f$.

A. *Stability of N-D linear discrete system associated with Kakeya's Theorem*

Here, we present stability issues associated with Kakeya's Theorems for N-dimensional (N-D, N≥1) discrete shift-invariant linear systems.

For $N = 1$ we have [2, p. 116]: if $1 > a_{n-1} > \cdots > a_1 > a_0 \geq 0$ then $R < 1$ and consequently $f(z)$ is stable.

For $N \geq 2$ we will sharpen Rudin's Theorem thus enabling us to apply Kakeya's Theorem when it is applicable, see [5]. For a proof of Rudin's Theorem see [5], where we generalized Rudin's Theorem that has a single 1-D condition and Strintzis's Theorem that has N 1-D conditions to have any number of up to N 1-D conditions. We let $f(z_1, z_2, ..., z_N)$ be the characteristic polynomial of an N-D ($N \geq 2$) dimensional linear shift-invariant discrete system. We say that the system is stable if

$$f(z_1, z_2, ..., z_N) \neq 0, |z_1| \leq 1, |z_2| \leq 1, ..., |z_N| \leq 1. \quad (57)$$

Rudin's Theorem states:

Theorem 4.4: $f(z_1, z_2, ..., z_N)$ is stable if and only if (=iff)

$$f(z_1, z_2, ..., z_N) \neq 0, |z_1| = 1, |z_2| = 1, ..., |z_N| = 1 \quad (58)$$

and

$$f(z, z, ..., z) \neq 0, |z| \leq 1. \quad (59)$$

The sharpened Rudin's theorem states:

Theorem 4.5:

$f(z_1, z_2, ..., z_N)$ is stable iff

$$f(z_1, z_2, ..., z_N) \neq 0, |z_1| = 1, |z_2| = 1, ..., |z_N| = 1 \quad (60)$$

and

$$f(z, z, ..., z) \neq 0, |z| < 1. \quad (61)$$

Proof.

Obviously, (60) implies $f(z, z, ..., z) \neq 0, |z| = 1$. Therefore, it suffices to test (61). ■

Now, we let $\phi(z)$ be the monic polynomial associated with $f(z, ..., z)$ and assume that its free coefficient is $\phi_0 \neq 0$. Next, we let $z = 1/Z$ and we define the monic polynomial $F(Z) := Z^{deg(\phi)}\phi(1/Z)/\phi_0$. Then, $f(z, ..., z) \neq 0, |z| < 1$ iff $F(Z) \neq 0, |Z| > 1$ iff $R_F \leq 1$. Hence, if $F(Z)$ satisfies one of the two Kakeya's conditions then $R_F \leq 1$ and (61) is satisfied.

V. Vieta based lower bounds on R and their application to stability

The proposed lower bounds that we will present here apply to real as well as complex polynomials.

We let $z_i, i = 1, ..., n$, $|z_1| \leq \cdots \leq |z_n|$ be the roots of $f(z)$, then [7]

$$\begin{aligned} f(z) &= z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \\ &= \prod_{i=1}^n (z - z_i) \\ &= z^n + (-1)^1 e_1 z^{n-1} + (-1)^2 e_2 z^{n-2} + ... + (-1)^k e_k z^{n-k} + ... + (-1)^n e_n, \end{aligned} \tag{62}$$

where

$$e_k := \sum_{(i_1,i_2,...i_k)\in Q_n^k} \prod_{\ell=1}^{k} z_{i_\ell} \tag{63}$$

are the elementary symmetric functions,

$$Q_{k,n} := \{(\alpha_1, ..., \alpha_k) \in \mathbb{N}^k : 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le n\}, \tag{64}$$

and its cardinality is

$$|Q_{k,n}| := \binom{n}{k}. \tag{65}$$

Hence, since $|z_i| \le R \ \forall i$ we obtain

$$|e_k| = |a_{n-k}| \le \binom{n}{k} R^k, \tag{66}$$

or

$$R \ \ge \underline{R} := \max_{k\in\{1,2,...,n\}} \left\{ |a_{n-k}| \binom{n}{k}^{-1} \right\}^{1/k}.$$

Remark 5.1:
Note that if $\underline{R} \ge 1$ there exists a root z_i of $f(z)$ such that $|z_i| \ge 1$, consequently $f(z)$ is not stable. This is a sufficient condition for instability of $f(z)$.

An upper bound on $r = |z_1|$ can be obtained as follows.
Assuming that $a_0 \ne 0$ we let

$$F(z) := a_0^{-1} z^n f(z^{-1}) \tag{67}$$

$$= \prod_{i=1}^{n} (z - z_i^{-1}) \tag{68}$$

$$= z^n + \frac{a_1}{a_0} z^{n-1} + \frac{a_2}{a_0} z^{n-2} + \cdots + \frac{a_k}{a_0} z^{n-k} + \cdots + \frac{a_{n-1}}{a_0} z + \frac{1}{a_0} \tag{69}$$

$$= z^n + (-1)^1 E_1 z^{n-1} + (-1)^2 E_2 z^{n-2} + ... + (-1)^k E_k z^{n-k} + ... + (-1)^n E_n, \tag{70}$$

where $z_i, i = 1, ..., n$ are the zeros of $f(z)$ and the E_k's are the following elementary symmetric functions of $F(z)$, i.e.,

$$E_k := \sum_{(i_1,i_2,...i_k)\in Q_n^k} \prod_{\ell=1}^{k} z_{i_\ell}^{-1}. \tag{71}$$

We have

$$|E_k| = \left|\frac{a_k}{a_0}\right| \le \binom{n}{k} \frac{1}{r^k}. \tag{72}$$

Hence,

$$r \le \left(\left|\frac{a_0}{a_k}\right| \binom{n}{k} \right)^{1/k}, \tag{73}$$

or

$$r \ \le \bar{r} \tag{74}$$

$$:= \min_{k\in\{1,...,n\}} \left\{ \left[\left|\frac{a_0}{a_k}\right| \binom{n}{k} \right]^{1/k} \right\}, \tag{75}$$

where $a_n := 1$, and we interpret division by zero as infinity.

Notice that if both $\underline{R}$ and $\bar{r}$ are attained at $k = n$ then $\bar{r} = \underline{R}$. Furthermore, for numerous simulated examples we always obtained $\bar{r} \le \underline{R}$.

By randomly choosing N polynomials of degree n and letting
$\mathbf{P}(i,j) := \#\{(i,j) : i = \arg\min(\bar{r}), j = \arg\max(\underline{R})\}/N$ we always obtained the following pattern for $\mathbf{P}$

$$\mathbf{P} = \begin{bmatrix} * & * & \cdots & * & * & 0 \\ * & * & \cdots & * & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & * \end{bmatrix}, \tag{76}$$

where * denotes a positive fraction.

Therefore, we conjecture that $\bar{r} \le \underline{R}$, or

$$\min_{k\in\{1,...,n\}} \left\{ \left[\left|\frac{a_0}{a_k}\right| \binom{n}{k} \right]^{1/k} \right\} \le \max_{k\in\{1,2,...,n\}} \left\{ |a_{n-k}| \binom{n}{k}^{-1} \right\}^{1/k}. \tag{77}$$

and $\mathbf{P}$ is as in (76).

Example 5.2: We carried out a million trials. In each trial we generated a random polynomial of degree 5 whose roots were uniformly distributed within $\{z \in \mathbb{C} : |z| \le 3\}$. We thus obtained

$$\mathbf{P} = \begin{bmatrix} 0.0045 & 0.0014 & 0.0017 & 0.0024 & 0 \\ 0.0015 & 0.0006 & 0.0004 & 0 & 0 \\ 0.005600 & 0.0024 & 0 & 0 & 0 \\ 0.0441 & 0 & 0 & 0 & 0.0 \\ 0 & 0 & 0 & 0 & 0.9355 \end{bmatrix}, \quad (78)$$

Example 5.3: In this example we report the following simulation results. For each n and $Rmax$ we generated 10000 polynomials of degree n whose roots were uniformly distributed in $\{z \in \mathbb{C} : |z| \le Rmax\}$.

TABLE IV
FRACTION OF UNSTABLE POLYNOMIALS DETECTED BY THE SUFFICIENT CONDITION $\underline{R} > 1$.

n	$Rmax = 1.5$	$Rmax = 2$	$Rmax = 2.5$	$Rmax = 3$	$Rmax = 3.5$
2	0.6030	0.8346	0.9291	0.9674	0.9841
3	0.4948	0.8184	0.9358	0.9741	0.9915
4	0.4276	0.8169	0.9488	0.9863	0.9939
5	0.3864	0.8311	0.9593	0.9904	0.9970
6	0.3691	0.8418	0.9687	0.9936	0.9987
7	0.3516	0.8566	0.9742	0.9964	0.9991
8	0.3231	0.8657	0.9813	0.9977	0.9992
9	0.3027	0.8809	0.9863	0.9982	0.9998
10	0.2940	0.8891	0.9887	0.9996	0.9998

Notice that the rows and columns are increasing, except $Rmax = 1.5$'s column which is decreasing.
for

Example 5.4: The study in this section led us to consider the following disturbing example. We will present this example by using Matlab's notation. If we choose a stable polynomial whose roots are z $= 0.1 : 0.01 : 0.9$ and compute a=poly(z), a=abs(**a**), ind=a $<$ eps, where eps=$2.2{\cdot}10^{-16}$ is Matlab's constant. It turned out that $n = 82$ and by observing ind we obtain that the 23 trailing values of a where less than eps. Hence, although $r \ge 0.1$, poly(z) contains 23 roots that are numerically zero.

VI. Conclusion

In this self contained article we reviewed known upper bounds on R, i.e., the norm-one bound R_1 and norm-infinity bound also called Montel's bound R_M. We presented new proofs based on Cauchy's proof technique that sharpened them. We removed the restriction that these bounds are at least one and showed how to apply them to 1-D stability problems. We improved these bounds for real polynomials by using linear programming. We reviewed Kakeya's Theorem, extended it, and showed how to apply it when it is applicable to test stability of N-D, $N \ge 2$, shift-invariant discrete linear systems by sharpening Rudin's Theorem. We obtained a lower bound $\underline{R}$ on R by using Vieta's formulas, applied it to obtain a sufficient condition for instability of 1-D systems, and on the way arrived at two conjectures. Future research may focus on (i) proving the proposed two conjectures; and, (ii) improving the spectral radius of matrices by combining their characteristic equation $f(z)$ with the β-method, or the LP-method, or both, see Remark 3.8.

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