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Research Paper

Research on Inverse Fractional Hyperbolic Functions

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ABSTRACT: *In this paper, inverse fractional hyperbolic function is studied. By usinga new multiplication of fractional analytic functions,the expressions of six inverse fractional hyperbolic functions are obtained. These expressions are presented in fractional logarithmic functions. In fact, these results are generalizations oftraditional calculus results.*
KEYWORDS: *Inverse Fractional Hyperbolic Functions, New Multiplication, Fractional Analytic Functions, Fractional Logarithmic Functions.*

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I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining the order of real number or complex number. Fractional calculus attracted the attention of great mathematicians, many of whom contributed directly or indirectly to its development. They include Euler, Laplace, Fourier, Abel, Riemann and Liouville. In the past decade, fractional calculus has been applied to almost every field of science, engineering and mathematics, such as mechanics, signal processing, robotics, electrical engineering,viscoelasticity, economics, bioengineering, and control [1-12].However, different from the traditional calculus, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definitions are Riemann-Liouville (R-L) fractional derivatives, Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [13-16].

This paper studies the inverse fractional hyperbolic function. The expressions of six inverse fractional hyperbolic functions are obtained by using a new multiplication of fractional analytic functions.And these expressions are presented in fractional logarithmic functions. In fact, these results we obtained are generalizations of ordinary calculus results.

II. PRELIMINARIES

Firstly,the fractional analytic function is introduced below.

Definition 2.1([17])**:**Letx, x_0, anda_k be real numbers for all k, $x_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, an$\alpha$-fractional power series on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic atx_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on$[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.2([18])**:**If$0<\alpha\leq 1$, and x_0 is a real number. Let$f_\alpha(x^\alpha)$ and$g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \quad (1)$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \quad (2)$$

Then we define

$$f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)$$

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$$= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (x-x_0)^{k\alpha}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) (x-x_0)^{k\alpha}. \quad (3)$$

Equivalently,

$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)$

$$= \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x-x_0)^{\alpha} \right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x-x_0)^{\alpha} \right)^{\otimes k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x-x_0)^{\alpha} \right)^{\otimes k}. \quad (4)$$

Definition 2.3: If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic function such that $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $[f_\alpha(x^\alpha)]^{\otimes -1}$.

In the following, the composition of two fractional analytic functions is introduced.

Definition 2.4 ([19])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x-x_0)^{\alpha} \right)^{\otimes k}, \quad (5)$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x-x_0)^{\alpha} \right)^{\otimes k}. \quad (6)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!} (g_\alpha(x^\alpha))^{\otimes k}, \quad (7)$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (f_\alpha(x^\alpha))^{\otimes k}. \quad (8)$$

Definition 2.5 ([19])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \quad (9)$$

Then $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.6 ([20])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)}$ is defined by

$$f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)} = E_\alpha\left(g_\alpha(x^\alpha) \otimes Ln_\alpha(f_\alpha(x^\alpha)) \right). \quad (10)$$

Some fractional analytic functions are introduced below.

Definition 2.7: Suppose that $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes k}. \quad (11)$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

In addition, the α-fractional hyperbolic sine function is

$$sinh_\alpha(x^\alpha) = \frac{1}{2}\left(E_\alpha(x^\alpha) - E_\alpha(-x^\alpha) \right). \quad (12)$$

And the α-fractional hyperbolic cosine function is

$$cosh_\alpha(x^\alpha) = \frac{1}{2}\left(E_\alpha(x^\alpha) + E_\alpha(-x^\alpha) \right). \quad (13)$$

The α-fractional hyperbolic tangent function is

$$tanh_\alpha(x^\alpha) = sinh_\alpha(x^\alpha) \otimes [cosh_\alpha(x^\alpha)]^{\otimes -1}. \quad (14)$$

And the α-fractional hyperbolic cotangent function is

$$coth_\alpha(x^\alpha) = cosh_\alpha(x^\alpha) \otimes [sinh_\alpha(x^\alpha)]^{\otimes -1}. \quad (15)$$

The α-fractional hyperbolic secant function is

$$sech_\alpha(x^\alpha) = [cosh_\alpha(x^\alpha)]^{\otimes -1}. \quad (16)$$

And the α-fractional hyperbolic cosecant function is

$$csch_\alpha(x^\alpha) = [sinh_\alpha(x^\alpha)]^{\otimes -1}. \quad (17)$$

In the following, we introduce inverse fractional hyperbolic function functions.

Definition 2.8: Let$0 < \alpha \leq 1$. Then $arcsinh_\alpha(x^\alpha)$ is the inverse function of$sinh_\alpha(x^\alpha)$, and it is called inverse α-fractional hyperbolic sine function. $arccosh_\alpha(x^\alpha)$ is the inverse function of $cosh_\alpha(x^\alpha)$, and we say that it is the inverse α-fractional hyperbolic cosine function. Moreover, $arctanh_\alpha(x^\alpha)$ is the inverse function of $tanh_\alpha(x^\alpha)$, and is called the inverse α-fractional hyperbolictangent function. $arccoth_\alpha(x^\alpha)$ is the inverse function of $coth_\alpha(x^\alpha)$, and it is called the inverse α-fractional hyperboliccotangent function. $arcsech_\alpha(x^\alpha)$ is the inverse function of $sech_\alpha(x^\alpha)$, and it is the inverse α-fractional hyperbolicsecant function. $arccsch_\alpha(x^\alpha)$ is the inverse function of $csch_\alpha(x^\alpha)$, and we say that it is the inverse α-fractional hyperboliccosecant function.

III. MAIN RESULT

In this section, the expressions of inverse fractional hyperbolic functions are obtained.

Theorem 3.1:*Let* $0 < \alpha \leq 1$, *then*

$$arcsinh_\alpha(x^\alpha) = Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + \left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} + 1\right)^{\otimes\frac{1}{2}}\right) \quad (18)$$

for all $\frac{1}{\Gamma(\alpha+1)}x^\alpha \in R$.

$$arccosh_\alpha(x^\alpha) = Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + \left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} - 1\right)^{\otimes\frac{1}{2}}\right) \quad (19)$$

for all $\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} > 1$.

$$arctanh_\alpha(x^\alpha) = \frac{1}{2}Ln_\alpha\left(\left(1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \otimes \left(1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right) \quad (20)$$

for all $\frac{1}{\Gamma(\alpha+1)}x^\alpha < 1$.

$$arccoth_\alpha(x^\alpha) = \frac{1}{2}Ln_\alpha\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha - 1\right)^{\otimes -1}\right) \quad (21)$$

for all $\frac{1}{\Gamma(\alpha+1)}x^\alpha > 1$ or$\frac{1}{\Gamma(\alpha+1)}x^\alpha < -1$.

$$arcsech_\alpha(x^\alpha) = Ln_\alpha\left(\left(1 + \left(1 - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right) \quad (22)$$

for all $0 < \frac{1}{\Gamma(\alpha+1)}x^\alpha \leq 1$.

$$arccsch_\alpha(x^\alpha)$$

$$= Ln_\alpha\left(\left(1 + \left(1 + \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right) \text{for all} \frac{1}{\Gamma(\alpha+1)}x^\alpha > 0. \quad (23)$$

$$= Ln_\alpha\left(\left(1 - \left(1 + \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right) \text{for all} \frac{1}{\Gamma(\alpha+1)}x^\alpha < 0. \quad (24)$$

Proof Since $sinh_\alpha(x^\alpha) = \frac{1}{2}\left(E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)\right)$, it follows that

$$sinh_\alpha\left(arcsinh_\alpha(x^\alpha)\right) = \frac{1}{2}\left(E_\alpha\left(arcsinh_\alpha(x^\alpha)\right) - E_\alpha\left(-arcsinh_\alpha(x^\alpha)\right)\right) = \frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (25)$$

Thus,

$$\left(\left(E_\alpha\left(arcsinh_\alpha(x^\alpha)\right)\right)^{\otimes 2}-1\right)\otimes\left(E_\alpha\left(arcsinh_\alpha(x^\alpha)\right)\right)^{\otimes -1}=\frac{2}{\Gamma(\alpha+1)}x^\alpha. \quad (26)$$

And hence,

$$\left(E_\alpha\left(arcsinh_\alpha(x^\alpha)\right)\right)^{\otimes 2}-2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes E_\alpha\left(arcsinh_\alpha(x^\alpha)\right)-1=0. \quad (27)$$

Therefore,

$$E_\alpha\left(arcsinh_\alpha(x^\alpha)\right)=\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+1\right)^{\otimes\frac{1}{2}}. \quad (28)$$

So,

$$arcsinh_\alpha(x^\alpha)=Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+1\right)^{\otimes\frac{1}{2}}\right).$$

Similarly, since $cosh_\alpha(x^\alpha)=\frac{1}{2}\left(E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)\right)$, we have

$$cosh_\alpha\left(arccosh_\alpha(x^\alpha)\right)=\frac{1}{2}\left(E_\alpha\left(arccosh_\alpha(x^\alpha)\right)+E_\alpha\left(-arccosh_\alpha(x^\alpha)\right)\right)=\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (29)$$

Hence,

$$\left(\left(E_\alpha\left(arccosh_\alpha(x^\alpha)\right)\right)^{\otimes 2}+1\right)\otimes\left(E_\alpha\left(arccosh_\alpha(x^\alpha)\right)\right)^{\otimes -1}=2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (30)$$

So,

$$\left(E_\alpha\left(arccosh_\alpha(x^\alpha)\right)\right)^{\otimes 2}-2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes E_\alpha\left(arccosh_\alpha(x^\alpha)\right)+1=0. \quad (31)$$

Thus,

$$E_\alpha\left(arccosh_\alpha(x^\alpha)\right)=\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-1\right)^{\otimes\frac{1}{2}}. \quad (32)$$

Therefore,

$$arccosh_\alpha(x^\alpha)=Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-1\right)^{\otimes\frac{1}{2}}\right).$$

On the other hand, since

$$tanh_\alpha(x^\alpha)=sinh_\alpha(x^\alpha)\otimes[cosh_\alpha(x^\alpha)]^{\otimes -1}=\left(E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)\right)\otimes[E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)]^{\otimes -1}. \quad (33)$$

It follows that

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$$=tanh_\alpha\left(arctanh_\alpha(x^\alpha)\right)$$

$$=\left(E_\alpha\left(arctanh_\alpha(x^\alpha)\right)-E_\alpha\left(-arctanh_\alpha(x^\alpha)\right)\right)\otimes\left[E_\alpha\left(arctanh_\alpha(x^\alpha)\right)+E_\alpha\left(-arctanh_\alpha(x^\alpha)\right)\right]^{\otimes -1}$$

$$=\left(\left(E_\alpha\left(arctanh_\alpha(x^\alpha)\right)\right)^{\otimes 2}-1\right)\otimes\left[\left(E_\alpha\left(arctanh_\alpha(x^\alpha)\right)\right)^{\otimes 2}+1\right]^{\otimes -1}. \quad (34)$$

Thus,

$$\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes\left(E_\alpha\left(arctanh_\alpha(x^\alpha)\right)\right)^{\otimes 2}=1+\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (35)$$

Therefore,

$$\left(E_\alpha\left(arctanh_\alpha(x^\alpha)\right)\right)^{\otimes 2}=\left(1+\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}. \quad (36)$$

So,

$$arctanh_\alpha(x^\alpha)=\frac{1}{2}Ln_\alpha\left(\left(1+\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right).$$

Similarly, since

$coth_\alpha(x^\alpha) = cosh_\alpha(x^\alpha)\otimes[sinh_\alpha(x^\alpha)]^{\otimes -1} = \left(E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)\right)\otimes[E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)]^{\otimes -1}$.(37)

We obatin

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$= coth_\alpha\left(arccoth_\alpha(x^\alpha)\right)$

$$= \left(E_\alpha\left(arccoth_\alpha(x^\alpha)\right) + E_\alpha\left(-arccoth_\alpha(x^\alpha)\right)\right)\otimes\left[E_\alpha\left(arccoth_\alpha(x^\alpha)\right) - E_\alpha\left(-arccoth_\alpha(x^\alpha)\right)\right]^{\otimes -1}$$

$$= \left(\left(E_\alpha\left(arccoth_\alpha(x^\alpha)\right)\right)^{\otimes 2} + 1\right)\otimes\left[\left(E_\alpha\left(arccoth_\alpha(x^\alpha)\right)\right)^{\otimes 2} - 1\right]^{\otimes -1}. \quad (38)$$

Hence,

$$\left(E_\alpha\left(arccoth_\alpha(x^\alpha)\right)\right)^{\otimes 2} = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha - 1\right)^{\otimes -1}. \quad (39)$$

Therefore,

$$arccoth_\alpha(x^\alpha) = \frac{1}{2}Ln_\alpha\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha - 1\right)^{\otimes -1}\right).$$

In addition, since

$$sech_\alpha(x^\alpha) = \left(cosh_\alpha(x^\alpha)\right)^{\otimes -1} = 2\left(E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)\right)^{\otimes -1}, \quad (40)$$

it follows that

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$$= sech_\alpha\left(arcsech_\alpha(x^\alpha)\right)$$

$$= 2\cdot\left[E_\alpha\left(arcsech_\alpha(x^\alpha)\right) + E_\alpha\left(-arcsech_\alpha(x^\alpha)\right)\right]^{\otimes -1}$$

$$= 2\cdot E_\alpha\left(arcsech_\alpha(x^\alpha)\right)\left[\left(E_\alpha\left(arcsech_\alpha(x^\alpha)\right)\right)^{\otimes 2} + 1\right]^{\otimes -1}. \quad (41)$$

Thus,

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\left(E_\alpha\left(arcsech_\alpha(x^\alpha)\right)\right)^{\otimes 2} - 2\cdot E_\alpha\left(arcsech_\alpha(x^\alpha)\right) + \frac{1}{\Gamma(\alpha+1)}x^\alpha = 0. \quad (42)$$

And hence,

$$E_\alpha\left(arcsech_\alpha(x^\alpha)\right) = \left(1 + \left(1 - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}. \quad (43)$$

So,

$$arcsech_\alpha(x^\alpha) = Ln_\alpha\left(\left(1 + \left(1 - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right).$$

Finally, since

$$csch_\alpha(x^\alpha) = \left(sinh_\alpha(x^\alpha)\right)^{\otimes -1} = 2\left(E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)\right)^{\otimes -1}. \quad (44)$$

It follows that

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$$= csch_\alpha\left(arccsch_\alpha(x^\alpha)\right)$$

$$= 2\cdot\left[E_\alpha\left(arccsch_\alpha(x^\alpha)\right) - E_\alpha\left(-arccsch_\alpha(x^\alpha)\right)\right]^{\otimes -1}$$

$$= 2 \cdot E_\alpha\left(arccsch_\alpha(x^\alpha)\right)\left[\left(E_\alpha\left(arccsch_\alpha(x^\alpha)\right)\right)^{\otimes 2} - 1\right]^{\otimes -1}. \quad (45)$$

And hence,

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes \left(E_\alpha\left(arccsch_\alpha(x^\alpha)\right)\right)^{\otimes 2} - 2 \cdot E_\alpha\left(arccsch_\alpha(x^\alpha)\right) - \frac{1}{\Gamma(\alpha+1)}x^\alpha = 0. \quad (46)$$

If $\frac{1}{\Gamma(\alpha+1)}x^\alpha > 0$, then

$$E_\alpha\left(arcsech_\alpha(x^\alpha)\right) = \left(1 + \left(1 + \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}. \quad (47)$$

If $\frac{1}{\Gamma(\alpha+1)}x^\alpha < 0$, then

$$E_\alpha\left(arcsech_\alpha(x^\alpha)\right) = \left(1 - \left(1 + \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}. \quad (48)$$

Q.e.d.

IV. CONCLUSION

In this paper, we obtain the expressions of six inverse fractional hyperbolic functions. These expressions are presented in fractional logarithmic functions.In addition, these results are generalizations of classical calculus results. A new multiplication of fractional analytic functions plays an important role in this article. In the future, we will extend ourresearch field to fractional differential equations and fractional calculus.

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