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Research Paper

A New Look At Monoid

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Abstract. *In this paper, the concepts of a Monoid in a generalised setting (multiset) is introduced. The study of the closure of multiset operations on the class of finite such structures is carried out. It is established that the root set of such a generalised monoid is a sub monoid. Further studies reveals that the multiset operation expressions of cancellable and commutative generalised monoids are cancellable and commutative respectively.*
Key Words: Monoid, Multiset, Multiset Operations, Closure, Cancellable and Commutative.

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I. Introduction.

Cantor is referred to as the father of set theory. In his cardinal key axiom he stated that an element must belong to a set only once. However, as research grows, his theories could not address so many fundamental issues such as the hydrogen molecules in water, DNA strands among other reasons. Thus the emergence of multiset (mset for short) which is a collection of objects in a set in which repetitions are allowed. For the various applications of msets the reader is refered to article [1], [4,], [7], [9], and [11]. It is observed from the survey of available literature on msets and applications that the idea of mset was hinted by R. Dedikind in 1888. The mset theory which generalizes set theory as a special case was introduced by Cerf et al.[2]. The term mset, as noted by Knuth [4] was first suggested by N.G de Bruijn in a private communication to him. Further study was carried out by Yager [14], Blizard [1]. Other researchers ([5], [7], [8]) gave a new dimension to the mset theory.

Msets has been established as a generalised version of set [1,7,9,14] . Several authors have studied the structures of the classical sets under the generalised settings, such as: mset topological space [10]. The introduction of the concepts of relations, function, composition, and equivalence in msets context. [3], Tella and Daniel have considered sets of mappings between msets and studied about group and symmetric groups under mset perspective. ([12], [13]) Nazmul et al. improved on Tella and Daniel's work and added two axioms [6]. In this paper we present the study of monoid in mset context while we lay more emphasis on the closure of mset operation on multi monoid and that of its homomorphism. From the literatures, there may be some variations in the definition of monoid depending on the point of view of the different authors. However, in this paper we consider definitions in [15] and [16].

In addition to this section, we present some preliminary definitions in section two to make the paper self-contained and some fundamental results are presented in section three while the entire paper is summarized in section four.

II. Preliminaries

2.1 Definitions and notations

Definition 2.1.1[15, 16]: Let S be a set and $\mu: S \times S \to S$ a binary operation that maps each ordered pair (x, y) of S to an element $\mu(x, y)$ of S. The pair (S, μ)(or just S, if there is no fear of confusion) is called a **groupoid**. The mapping μ is called the product of (S, μ). We shall mostly write simply xy Instead of $\mu(x, y)$. If we want to emphasize the place of the operation then we often write $x.y$. The element $xy(= \mu(x, y))$ is the product of x and y in S.

Definition 2.1.2[15, 16]: A groupoid S is a Semigroup if the operation μ is associative; for all $x\mu(y\mu z) = (x\mu y)\mu z$. Thus a semigroup is a pair (S, μ) where S is a non empty set and μ is its binary operation on μ which satisfied two axioms:

(i) The closure property
(ii) The associativity property.

Definition 2.1.3[15,16](Monoid): Let S(that is (S, μ)) be a semigroup. An element $x \in S$ is a left identity of S if $y \in S: x\mu y = y$. Similarly, x is right identity of S if $\forall\, y \in S: y\mu x = y$. If x is both left and right identity of S, then x is called an identity of S. A semigroup is a **monoid** if it has an identity.

Definition 2.1.4[1]. An mset A over the set X can be defined as a function $C_A: X \to \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{0,1,2, \dots\}$ where the value $C_A(x)$ denote the number of times or multiplicity or count function of $x \; in \, A$. For example, Let $A = [x, x, x, y, y, y, z, z]$, then $C_A(x) = 3, C_A(y) = 3, C_A(z) = 2$. $[C_A(x) = 0 \Leftrightarrow x \notin A]$. If $C_M(x) = 0$ for all $x \in X$. We denote the empty mset by $\emptyset$. Then $C_\emptyset(x) = 0, \forall\, x \in X$. ($C_A(x) > 0 \Leftrightarrow x \in A$)..If $C_A(x) = n$ then the membership of x in A can be denoted by $x \in^n A$, meaning x belong to A exactly n times.

Presentation of mset on paper work became a challenged as every researcher has his thought in that aspect. However the use of square brackets was adopted in ([1], [9],[11]) to represent an mset and ever since then it has become a standard. For example if the multiplicity of the elements x, y and z in an mset M are 2,3 and 2 respectively, then the mset M can be represented as $M = [x, x, y, y, y, z, z,]$, others put it like $[x, y, z]_{2,3,2}$ or $[x^2, y^3, z^2]$ or $[x2, y3, z2]$ or $[2/x, 3/y, 2/z]$ depending on one's taste or expediencies. But for conveniences sake, curly bracket can be used instead of the square bracket.

Definition 2.1.5[1]: The cardinality of a mset M denoted $|M|$ or $card(M)$ is the sum of all the multiplicities of its elements given by the expression $|M| = \sum_{x \in X} c_A(x)$

Note: An mset M is said to be finite if $|M| < \infty$.

We denote the class of all finite msets A over the set X by $M(X)$.

Definition 2.1.6[2]: Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X given by $M^* = \{x \in X: C_M(x) > 0\}$. M^* is also called root set.

Definition 2.1.7[1](Equal msets): Two msets $A, B \in M(X)$ are said to be equal, denoted $A = B$ if and only if for any objects $x \in X, C_A(x) = C_B(x)$. This is to say that $A = B$ if the multiplicity of every element in A is equal to its multiplicity in B and conversely.

Definition 2.1.8[1]: The exact multiplicity axiom: $\forall x \forall y \forall n \forall m(x \in^n y \wedge x \in^m y) \to n = m$. In other words, the multiplicity with which an element belongs to a mset is unique.

Definition 2.1.9[1] (Submultiset): Let $A, B \in M(X)$. A is a submultiset (submset for short) of B, denoted by $A \subseteq B \; or \; B \supseteq A$, if $C_A(x) \leq C_B(x) \; for \; all \; x \in X$. Also if $A \subseteq B \; and \; A \neq B$, then A is called proper submset of B denoted by $A \subset B$. In other words $A \subset B$ if $A \subseteq B$ and there exist at least an $x \in X$ such that $C_A(x) < C_B(x)$. We assert that an mset B is called the parent mset in relation to the mset A.

Note that: For any two msets $A, B \in M(X)$, $A = B$ if and only if $A \subseteq B$ and. $B \subseteq A$.

Definition. 2.1.10[1]: (Regular or Constant mset): An mset$A$ over the set X is called regular or constant if all its elements are of the same multiplicities, i.e for any $x, y \in A$,

$x \neq y \Rightarrow C_A(x) = C_A(y)$.

Definition 2.1.11: [9] ($\wedge$ and $\vee$ notations): The notations $\wedge$ and $\vee$ denote the minimum and maximum operator respectively, for instance;

$C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\} \; and \; C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

2.2 Multiset operations.

Definition 2.2.1[9] (msets union): Let $A, B \in M(X)$. The union of $A \; and \; B$ denoted $A \cup B$ is the mset defined by $C_{A \cup B}(x) = \{C_A(x) \vee C_B(x)\} \; \forall\, x \in X$,

Definition 2.2.2[9] (msets intersection) Let$A, B \in M(X)$.The intersection of two mset A and B denoted by $A \cap B$, is the mset for which

$C_{A \cap B}(x) = \{C_A(x) \wedge C_B(x)\} \; \forall\, x \in X$.

Definition 2.2.3[9] (mset addition): Let $A, B \in M(X)$. The direct sum or arithmetic addition of A and B denoted by $A + B$ or $A \uplus B$ is the mset defined by

$C_{A+B}(x) = C_A(x) + C_B(x) \forall\, x \in X$.

Note that $|A \uplus B| = |A \cup B| + |A \cap B|$.

Definition 2.2.4[9] (mset difference): Let $A, B \in M(X)$, then the difference of B from A, denoted by $A - B$ is the mset such that $C_{A-B}(x) = \big(C_A(x) - C_B(x)\big) \vee 0, \forall\, x \in X$. If $B \subseteq A$, then

$C_{A-B}(x) = C_A(x) - C_B(x)$.

It is sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$ this definition still holds. It follows that the deletion of an element x from an mset A give rise to a new mset $A' = A - x$ such that $C_{A'}(x) = (C_A(x) - 1) \vee 0$.

Definition 2.2.5[8] (mset symmetric difference): Let X be a set and $A, B \in M(X)$Then the symmetric difference, denoted $A \Delta B$, is defined by $C_{A \Delta B}(x) = |C_A(x) - C_B(x)|$.

Note that $A \Delta B = (A - B) \cup (B - A)$.

Definition 2.2.6[8] (mset complement): Let $G = \{A_1, A_2, \dots, A_n\}$ be a family of finite msets generated from the set X. Then, the maximum mset Z is defined by $C_Z(x) = max_{A \in G} C_A(x)$ for all $A \in G$ and $x \in X$. The Complement of an mset A, denoted by $\overline{A}$, is defined:

$\bar{A} = Z - A$ *such that* $C_{\bar{A}}(x) = C_Z(x) - C_A(x), for\ all\ x \in X.$
Note that $A_i \subseteq Z\ for\ all\ i.$
Definition 2.2.7[8] (Multiplication by Scalar): Let $A \in M(X)$, then the scalar multiplication denoted by $b.A$ is defined by $C_{b.A}(x) = b.C_A(x),\ and\ b \in \{1,2,3, \dots\}$.
Definition 2.2.8[8] (Arithmetic Multiplication): Let $A, B \in M(X)$, then the Arithmetic Multiplication denoted by $A.B$ is defined by $C_{A.B}(x) = C_A(x).C_B(x)\ \forall\ x \in X.$
Definition 2.2.9[7] (Raising to an Arithmetic Power): Let $A \in M(X)$, then A raised to power n denoted by A^n **is** defined:
$C_{A^n}(x) = \left(C_A(x)\right)^n\ for\ n \in \{0,1,2,3, \dots\}$.
Theorem 2.2.10[19]: Let X be a set and let $A \in M(X)$. Then
(i) $A^* = A^0$.
(ii) $A^n . A^m = A^{n+m}$, and
(iii) $(A.B)^n = A^n . B^n\ for\ any\ n, m \in \{0,1,2, \dots\}$
Theorem 2.2.11[19]: For any $A \neq \emptyset$ such that $A \in M(X)$, then $(A^n)^* = A^*$ for $n \in \{0,1,2 \dots\}$
Definition **2.3.12[19]**: Let X be a groupoid, and $A \in M(X)$. A is said to be a multi-groupoid (mgroupoid for short) if the following condition is satisfied.
$C_A(xy) \geq C_A(x) \wedge C_A(y), \forall\ x, y\ \in X.$
We denote the class of all finite mgroupoids over X by $MGP(X)$.
Proposition 2.3.13: Let $A \in MGP(X)$. Then
(i) A^* is a sub groupoid of X.
(ii) $A^* \in MGP(X)$.
Definition 2.3.14[19]: Let $A \in MGP(X)$, then A is said to be a commutative mgroupoid if
$C_A(xy) = C_A(yx)\ \ \forall\ x, y \in X.$
Commutative mgroupoid can also be called Abelian mgroupoid.
Definition 2.3.15[19]: Let $A \in MGP(X)$ an element $a \in A$ is said to be cancellable if
$C_A(ax) = C_A(ay),\ and\ C_A(xa) = C_A(ya)$, implies $C_A(x) = C_A(y)$.
Definition 2.3.16[19]: Let $A \in MGP(X)$. Then A is said to be cancellable if a is cancellable for all $a \in A$..
Definition 2.3.17[20]: Let mset $A \in MGP(X)$, then A is said to be a semi –multigroup (semi-mgroup for short) if X is a semi-group.
We denote the class of all finite semi-mgroups over X by *SMG(X).*
Clearly $SMG(X) \subset MGP(X)$.
Theorem 2.3.18[20]: Let X be a semi-group and $A \in SMG(X)$. Then
(i) A^* is a sub semi-group of X and
(ii) $A^* \in SMG(X)$
Theorem 2.3.19[20]: Let X be a semi-group and let $A, B \in SMG(X)$, Then
(i) $A \cap B \in SMG(X)$.
(ii) $k.A \in SMG(X), k \in \{1,2 \dots .\}$
(iii) $A.B \in SMG(X)$
(iv)$A^n \in SMG(X), n \in \{0,1,2, \dots\}$
(v) $AoB \in SMG(X)$
Definition 2.3.20[20]: Let $A \in SMG(X)$ and let B be a submset of A. Then B can be said to be a sub mgroupoid of A, if $B \in SMG(X)$.
Definition 2.3.21[20]: Let $A \in SMG(X)$ an element $a \in A$ is said to be cancellable semi-mgroup if it is cancellable mgroupoid.
Definition 2.3.22[20]: Let $A \in SMG(X)$. Then A is said to be cancellable semi-mgroup if it is cancellable mroupoid
We denote the class of finite cancellable semi-mgroup as $\mathbb{C}SMG(X)$. That is,
$\mathbb{C}SMG(X) = \{A \in SMG(X)/A$ is cancellable$\}$.
Definition 2.3.23[20]: Let $A \in SMG(X)$, then A is said to be a commutative semi-mgroup if it is a commutative mgroupoid.
Commutative semi- mgroup can also be called Abelian semi-mgroup.
We denote the class of finite commutative semi-mgroup as $CSMG(X)$. That is,
$CSMG(X) = \{A \in SMG(X)/A$ is commutative$\}$.
Theorem 2.3.24[20]: Let $A, B \in SMG(X)$ such that A and B are commutative and cancellative. Then the following expressions are commutative and cancellative:
(i) $A \cap B$
(ii) $A \cup B$
(iii) $A + B$

(iv) $A - B$
(v) $A\Delta B$
(vi) $A \cdot B$
(vii) $kA, k \in \{1,2,3, \dots\}$
(viii) $A^n,\ n \in \{0,1,2, \dots\}$
(ix) AoB
Theorem 2.3.25[20]: Let $A, B \in \mathbb{C}SMG(X)$ and $A, B \in CSMG(X)$. Then the following are both satisfied
(i) $A \cap B \in \mathbb{C}SMG(X)$ and $A \cap B \in CSMG(X)$.
(ii) $k.A \in \mathbb{C}SMG(X)$ and $k.A \in CSMG(X)$, $k \in \{1,2 \dots.\}$
(iii) $A.B \in \mathbb{C}SMG(X)$ and $A.B \in CSMG(X)$.
(iv) $A^n \in \mathbb{C}SMG(X)$ and $A^n \in CSMG(X), n \in \{0,1,2, \dots\}$
(v) $AoB \in \mathbb{C}SMG(X)$ and $AoB \in CSMG(X)$
Theorem 2.3.26[20]: $\mathbb{C}SMG(X) = CSMG(X)$.

III. mset monoid.

Definition 3.1.1: An mset $A \in SMG(X)$ is said to be a multimonoid (mmonoid for short). If
(i) X is a monoid and
(ii) $C_A(e) \geq C_A(x) \forall\ x \in X$.
Where e is the identity element in X.
For example, the mset $A = \{0,1,2\}_{3,2,1}$ over the given additive semigroup $X = Z_3 = \{0,1,2\}$ is an mmonoid.
Let the class of all finite mmonoids over the monoid X be denoted by $MM(X)$
such that $A \neq \emptyset\ \forall\ A \in MM(X)$
Definition 3.1.2: Let $A \in MM(X)$ and let B be a submset of A. Then B can be said to be a sub mmonoid of A, if $B \in MM(X)$.
Definition 3.1.3; Composition of mmonoid: Let $A, B \in MM(X)$, then we call
$A \circ B$ the composition defined

$$C_{A \circ B}(x) = \vee\{C_A(y) \wedge C_B(z) : y, z \in X \ni yz = x\}$$

Theorem 3.1.4: $MM(X) \subset SMG(X)$.
Proof: It is straight forward from definition 3.1.1
Remark: It is pertinent to note that just as from the classical point of view,
$MM(X) \subset SMG(X) \subset MGP(X)$.
Proposition 3.1.5: Let X be a monoid and $A \in MM(X)$. Then A^* is a sub monoid of X.
Proof: Let $A \in MM(X)$, from the above remark $A \in MGP(X)$ and A^* is a subgroupoid of the groupoid X(theorem 2.3.13).
But since $A^* \subseteq X$ and X is a semi-group, we have A^* a semi-group. (1)
Now we show that $e \in A^*$. That is since $C_A(e) \geq C_A(x), \forall\ x \in X$(by hypothesis).
$C_A(x) > 0,$ for some $x \in X$, since $A \neq \emptyset$
Thus for such x, we have $C_A(e) \geq C_A(x) > 0$.
In particular $C_A(e) > 0$ i.e $e \in A^*$ (2)
Now from (1) and (2) above we have A^* a submonoid.
Proposition 3.1.6: Let $A \in MM(X)$, then $A^* \in MM(X)$.
Proof: Let $A \in MM(X)$. We need show that $A^* \in MM(X)$. But $A \in MGP(X)$(theorem 2.3.13)
Thus $A^* \in MGP(X)$(see [19]). But X is a semi-group. We have $A^* \in SMG(X)$ (1)
Now since $x \in A$, $C_A(x) > 0$ and $C_A(e) \geq C_A(x)\ \forall\ x \in X$ (by hypothesis)
Since $e \in A^*$ (proposition 3.1.5)
Then $C_{A^*}(e) = 1$. But for all $x \in X$, $C_{A^*}(x) = \{0,1\}$
Thus $C_{A^*}(e) \geq C_{A^*}(x)$ is valid
In particular $A^* \in MM(X)$.

3.2 Closure of mset operations on mmonoid.

Proposition 3.2.1: Let $A, B \in MM(X)$, Then
(i) $A \cap B \in MM(X)$.
(ii) $k.A \in MM(X)$, $k \in \{1,2 \dots.\}$
(iii) $A.B \in MM(X)$
(iv) $A^n \in MM(X), n \in \{0,1,2, \dots\}$
(v) $AoB \in MM(X)$
Proof: (i) Given that $A, B \in MM(X)$, then $A, B \in SMG(X)$
and $A \cap B \in SMG(X)$ (theorem 2.3.19).
Now we show that $C_{A\cap B}(e) \geq C_{A\cap B}(x)\ \forall\ x \in X$.

And $C_{A\cap B}(e) = C_A(e) \wedge C_B(e)$(by definition)
But $C_A(e) \geq C_A(x)\ \forall\ x \in X$ and $C_B(e) \geq C_B(x)\ \forall\ x \in X$. (by hypothesis)
Therefore $C_{A\cap B}(e) = C_A(e) \wedge C_B(e) \geq C_A(x) \wedge C_B(x) = C_{A\cap B}(x)$.
Thus $C_{A\cap B}(e) \geq C_{A\cap B}(x)\ \forall\ x \in X$.
In particular, $A \cap B \in MM(X)$.
(ii) Given that $A \in MM(X), k \in \{1,2,3, \dots\}$, then $A \in SMG(X)$
and $k.A \in SMG(X)$ (theorem 2.3.19).
Now we show that $C_{k.A}(e) \geq C_{k.A}(x)\ \forall\ x \in X$.
But $C_{k.A}(e) = k.C_A(e)$(by definition)
And $C_A(e) \geq C_A(x)\ \forall\ x \in X$, (by hypothesis)
Therefore $C_{k.A}(e) = k.C_A(e) \geq k.C_A(x) = C_{k.A}(x)$.
Thus $C_{k.A}(e) \geq C_{k.A}(x)\ \forall\ x \in X$.
In particular, $k.A \in MM(X)$.
(iii) Given that $A, B \in MM(X)$, then $A, B \in SMG(X)$
and $A.B \in SMG(X)$ (theorem 2.3.19).
Now we show that $C_{A.B}(e) \geq C_{A.B}(x)\ \forall\ x \in X$.
And $C_{A.B}(e) = C_A(e).C_B(e)$(by definition)
But $C_A(e) \geq C_A(x)\ \forall\ x \in X$ and $C_B(e) \geq C_B(x)\ \forall\ x \in X$. (by hypothesis)
Therefore $C_{A.B}(e) = C_A(e).C_B(e) \geq C_A(x).C_B(x) = C_{A.B}(x)$.
Thus $C_{A.B}(e) \geq C_{A.B}(x)\ \forall\ x \in X$.
In particular, $A.B \in MM(X)$.
(iv) Given that $A \in MM(X)$, then $A \in SMG(X)$
and $A^n \in SMG(X)$ (theorem 2.3.19).
Now we show that $C_{A^n}(e) \geq C_{A^n}(x)\ \forall\ x \in X$.
But $C_{A^n}(e) = (C_A(e))^n$(by definition)
But $C_A(e) \geq C_A(x)\ \forall\ x \in X$. (by hypothesis)
Therefore $C_{A^n}(e) = (C_A(e))^n \geq (C_A(x))^n = C_{A^n}(x)$.
Thus $C_{A^n}(e) \geq C_{A^n}(x)\ \forall\ x \in X$.
In particular, $A^n \in MM(X)$.
(v) Given that $A, B \in MM(X)$, then $A, B \in SMG(X)$
and $AoB \in SMG(X)$ (theorem 2.3.19).
Now we show that $C_{AoB}(e) \geq C_{AoB}(x)\ \forall\ x \in X$. And
$C_{AoB}(e) = \vee\{C_A(e) \wedge C_B(e), \text{since } e.e = e\}$.
since $C_A(e) \geq C_A(x)$ and $C_B(e) \geq C_B(x)\ \forall\ x \in X$(by hypothesis)
Thus $C_{AoB}(e) = \vee\{C_A(e) \wedge C_B(e)\} \geq \vee\{C_A(x) \wedge C_B(x)\} = C_{AoB}(e)$.
Therefore $C_{AoB}(e) \geq C_{AoB}(x)\ \forall\ x \in X$.
In particular, $AoB \in MM(X)$.
Remark: Let $A, B \in MM(X)$, Then $A \cup B,\ A + B,\ A - B,\ A\ \Delta\ B,\ \hat{A}$ need not be an mmonoid.

3.3 Cancellability and Commutativity of mmoniod.

Definition 3.3.1: Let $A \in MM(X)$. Then A is said to be cancellable mmonoid if it is cancellable semi-mgroup. We denote the class of finite cancellable mmonoid as $\mathbb{C}MM(X)$.
That is $\mathbb{C}MM(X) = \{A \in MM(X) \mid A \text{ is cancellable}\}$.

Definition 3.3.2: Let $A \in MM(X)$, then A is said to be a commutative mmonoid if commutative semi-mgroup. Commutative mmonoid can also be called Abelian semi-mgroup.

Example:**3.3.3:** Let $X = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e^2 = e\ and\ ab = ba = c$, $ac = ca = b, bc = cb = a$. Where e is the identity element. If $A = \{e, a, b, c\}_{3,2,3,2}$ is an mset over X. Clearly A is a commutative mmonoid.
We denote the class of finite commutative mmonoid as $CMM(X)$.
That is $CMM(X) = \{A \in MM(X) \mid A \text{ is commutative}\}$
Note that since $\mathbb{C}SMG(X) = CSMG(X)$.(Theorem 2.3.26), and if X is monoid then clearly $\mathbb{C}MM(X) = CMM(X)$. We denote $\mathfrak{C}MM(X) = \mathbb{C}MM(X) = CMM(X)$, the class of both cancellative and commutative mmonoid.

Proposition 3.3.4: Let $A \in MM(X)$, then $A \in CMM(X)$ If X is commutative.
Proof: Let $A \in MM(X)$ and X be a commutative mmonoid, then for all $x, y \in X$.
We have $xy = yx$.
Thus $C_A(xy) = C_A(yx)$. (uniqueness of multiplicities)(Definition 2.1.8)

3.4 mset operation on commutative and cancellable mmonoids.

Proposition 3.4.1: Let $A, B \in CMM(X)$ and $\mathbb{C}MM(X)$. Then the following expressions are both commutative and cancellative.

(i) $A \cap B$
(ii) $A \cup B$
(iii) $A + B$
(iv) $A - B$
(v) $A \Delta B$
(vi) $A \cdot B$
(vii) $kA, k \in \{1,2,3, \dots\}$
(viii) $A^n,\ n \in \{0,1,2, \dots\}$
(ix) AoB

Proof: From (i) to (x) The expressions are all commutative and cancellative (theorem 2.3.24 and proposition 3.2.1 and the fact that X is a monoid)

Proposition 3.4.2: Let $A, B \in \mathfrak{C}MM(X)$. Then
(i) $A \cap B \in \mathfrak{C}MM(X)$.
(ii) $k.A \in \mathfrak{C}MM(X), k \in \{1,2 \dots.\}$
(iii) $A.B \in \mathbb{C}MM(X)$
(iv) $A^n \in \mathfrak{C}MM(X), n \in \{0,1,2, \dots\}$
(v) $AoB \in \mathfrak{C}MM(X)$

Proof:
(i) Since $A, B \in \mathfrak{C}MM(X)$, then $A, B \in MM(X)$ (by definition) and
$A \cap B \in MM(X)$ (Proposition 3.2.1 (i))
But $A \cap B \in \mathbb{C}MM(X)$ (Theorem 2.3.19)
And $\mathfrak{C}MM(X) = CMM(X) = \mathbb{C}MM(X)$
Thus $A \cap B \in \mathfrak{C}MM(X)$
(ii) Since $A \in \mathfrak{C}MM(X)$, then $A \in MM(X)$ (by definition) and
$kA \in MM(X)$ (Proposition 3.2.1 (ii))
But $kA \in \mathfrak{C}MM(X)$ (Theorem 2.3.19)
And $\mathfrak{C}MM(X) = CMM(X) = \mathbb{C}MM(X)$
Thus $k.A \in \mathfrak{C}MM(X),\ k \in \{1,2 \dots.\}$
(iii) Since $A, B \in \mathfrak{C}MM(X)$, then $A, B \in MM(X)$ (by definition) and
$A.B \in MM(X)$ (Proposition 3.2.1 (iii))
But $A.B \in \mathfrak{C}MM(X)$ (Theorem 2.3.19)
And $\mathfrak{C}MM(X) = CMM(X) = \mathbb{C}MM(X)$
Thus $A.B \in \mathfrak{C}MM(X)$
(iv) Since $A \in \mathfrak{C}MM(X)$, then $A \in MM(X)$ (by definition) and
$A^n \in MM(X)$ (Proposition 3.2.1 (iv))
But $A^n \in \mathfrak{C}MM(X)$ (Theorem 2.3.19)
And $\mathfrak{C}MM(X) = CMM(X) = \mathbb{C}MM(X)$
Thus $A^n \in \mathfrak{C}MM(X), n \in \{0,1,2 \dots.\}$
(v) Since $A, B \in \mathfrak{C}MM(X)$, then $A, B \in MM(X)$ (by definition) and
$AoB \in MM(X)$ (Proposition 3.2.1 (v))
But $AoB\ \mathfrak{C}MM(X)$ (Theorem 2.3.19)
And $\mathfrak{C}MM(X) = CMM(X) = \mathbb{C}MM(X)$
Thus $AoB \in \mathfrak{C}MM(X)$

Note that $A \cup B$, $A + B$, $A - B$, $A\Delta B$, and $\hat{A}$ even though satisfied all the axioms buy need not be cancellable and commutative mmonoids since $A \cup B$, $A + B$, $A - B$, $A\Delta B$, and $\hat{A}$ are not mmonoids(theorem 2.3.14)

IV. Conclusion.

We have introduced and studied the concepts of monoid in multiset context (Multi Monoid). In the study, we have established the closure of some multiset operations over the class of finite multi monoids (mmonoid). We have established that the root set of an mmonoid is a sub monoid and sub mmonoid. Further studies reveals that the multiset operation expressions of cancellable and commutative generalised monoids are cancellable and commutative respectively while the closure of it need not hold on some operations in general.

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